



A primer on Topological Matter

Luis E. F. Foà Torres

Departamento de Física, FCFM, Universidad de Chile

All the material from these lectures will be here:





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driven systems

many-body interactions and decoherence

quantum transport

light-matter interaction

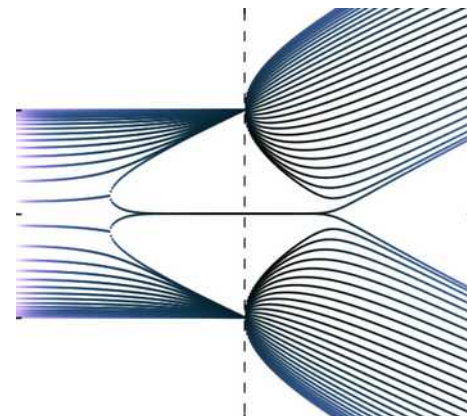
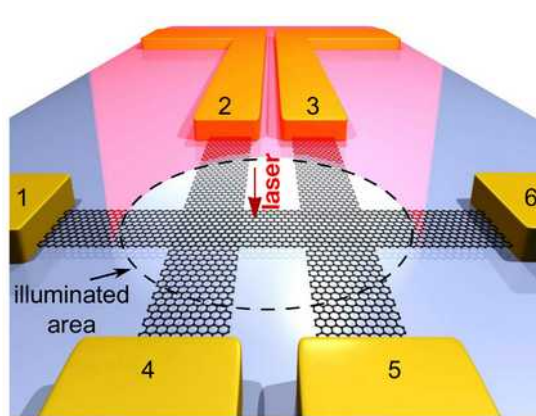
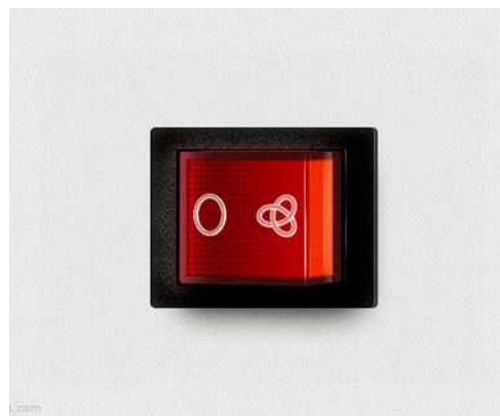
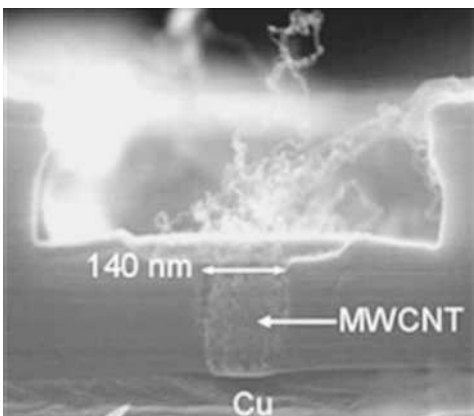
2D materials

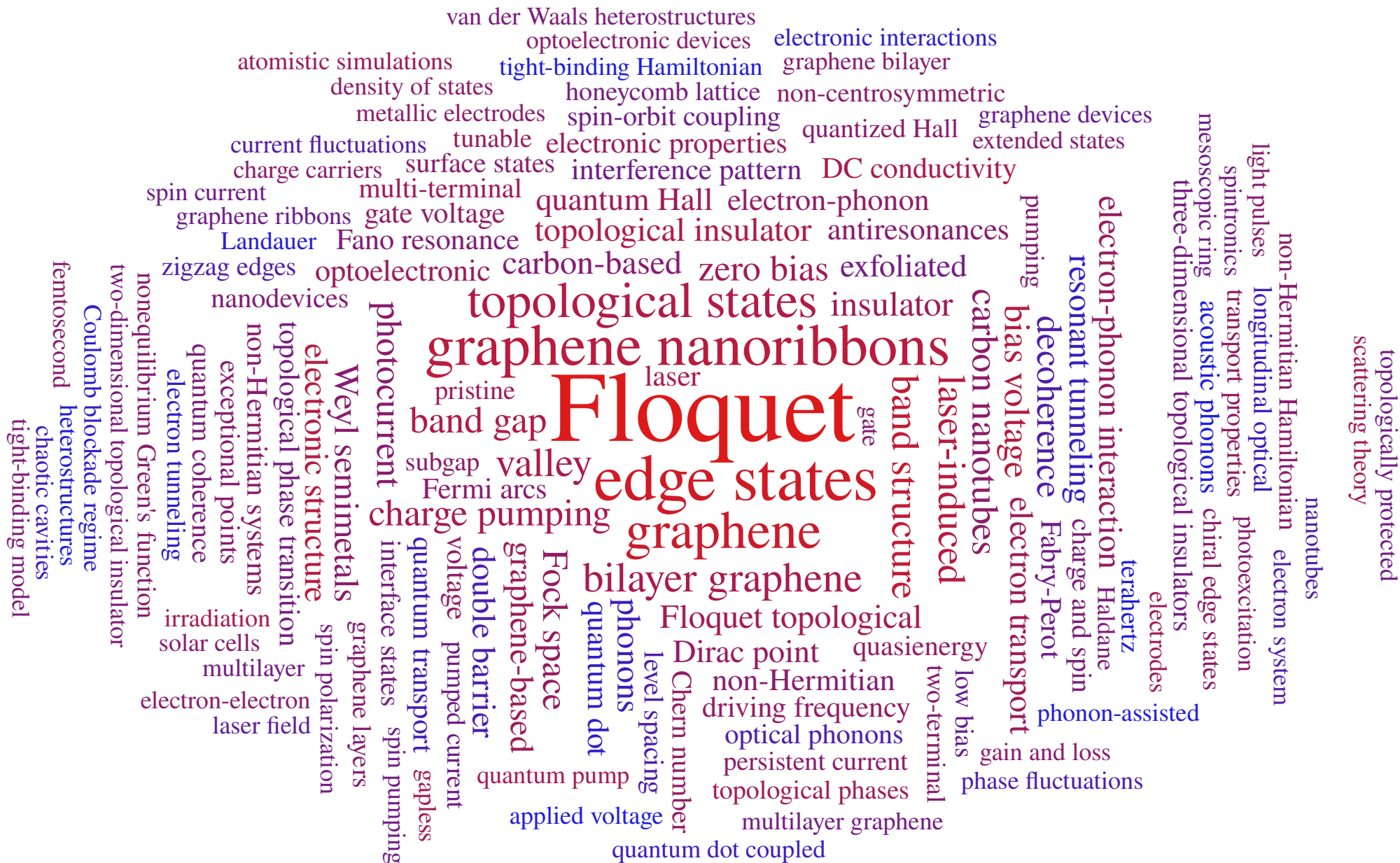
topological insulators

quantum materials

nanostructures

electron-phonon interaction





● Recently used
● Not recently used

<https://doi.org/10.1038/s41567-025-02888-8>

topologically protected scattering theory

Engineering Floquet Moiré Patterns for Scalable Photocurrents

Hernán L. Calvo, Luis E. F. Foa Torres, and Matias Berdakin*

Cite This: *Nano Lett.* 2025, 25, 1630–1636

Read Online

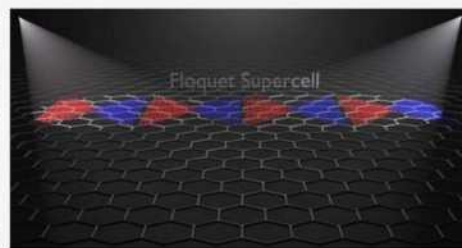
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Supporting Information

ABSTRACT: While intense laser irradiation and moiré engineering have independently proven powerful for tuning material properties on demand in condensed matter physics, their combination remains unexplored. Here we exploit tilted laser illumination to create spatially modulated light–matter interactions, leading to two striking phenomena in graphene. First, using two lasers tilted along the same axis, we create a quasi-1D supercell hosting a network of Floquet topological states that generate controllable and scalable photocurrents spanning the entire irradiated region. Second, by tilting lasers along orthogonal axes, we establish a 2D polarization moiré pattern giving rise to closed orbital propagation of Floquet states, reminiscent



Laser-Induced Spin Precession in Topological Insulator Devices

Esteban A. Rodríguez-Mena, Matías Berdakin,* and Luis E. F. Foa Torres*

Cite This: <https://doi.org/10.1021/acs.nanolett.5c03586>

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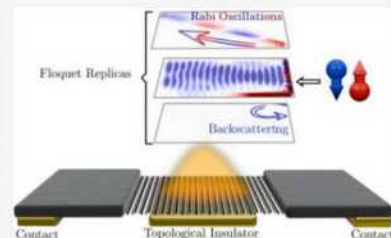
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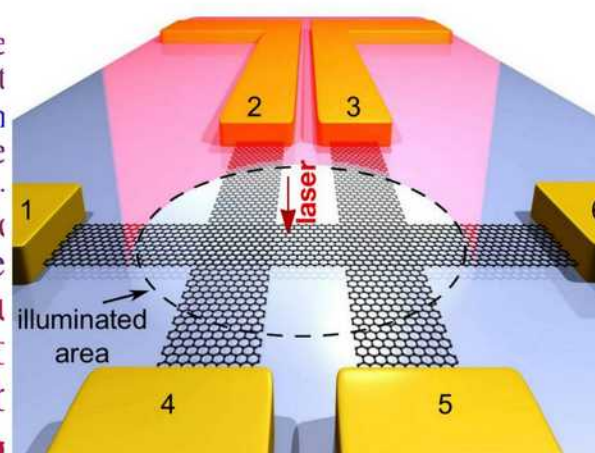
Supporting Information

ABSTRACT: Controlling spin currents in topological insulators (TIs) is crucial for spintronics but challenged by the robustness of their chiral edge states, which impedes the spin manipulation required for devices such as spin-field effect transistors (SFETs). We theoretically demonstrate that this challenge can be overcome by synergistically applying circularly polarized light and gate-tunable Rashba spin–orbit coupling (rSOC) to a 2D TI. Laser irradiation provides access to Floquet sidebands where rSOC induces controllable spin precession, leading to the generation of one-way, switchable spin-polarized photocurrents, a non-equilibrium effect forbidden in TIs under time-reversal symmetric (equilibrium) conditions. This mechanism effectively realizes SFET functionality within a driven TI, specifically operating within a distinct Floquet replica, offering a new paradigm for light-based control in topological spintronics.

KEYWORDS: spin photocurrents, topological insulators, two-dimensional materials, spin–orbit coupling



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edge states
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Dana Su

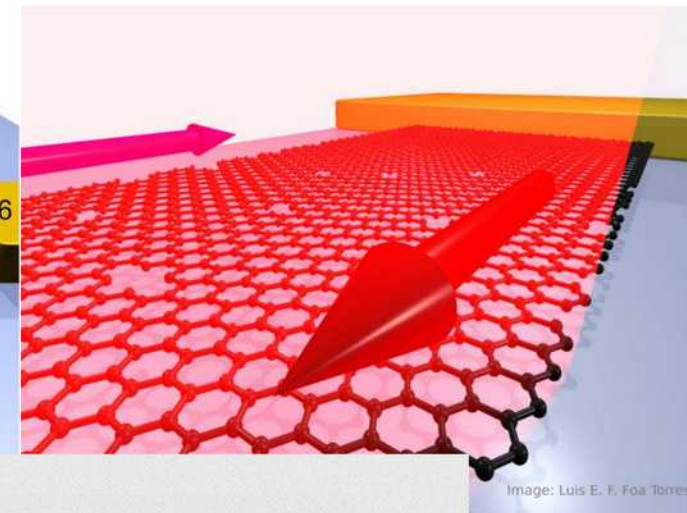


Image: Luis E. F. Foa Torres

topologically pr
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nature communications



Article

<https://doi.org/10.1038/s41467-025-57335-2>

Signatures of Floquet electronic steady states in graphene under continuous-wave mid-infrared irradiation

Received: 14 February 2024

Accepted: 19 February 2025

Published online: 28 February 2025

Check for updates

Yijing Liu^{1,9}, Christopher Yang^{2,9}, Gabriel Gaertner³, John Huckabee³, Alexey V. Suslov⁴, Gil Refael², Frederik Nathan⁵, Cyprian Lewandowski^{4,6}, Luis E. F. Foa Torres⁷, Iliya Esin^{2,8}✉, Paola Barbara¹✉ & Nikolai G. Kalugin³✉

Light-induced phenomena in materials can exhibit exotic behavior that extends beyond equilibrium properties, offering new avenues for under-

Many (unexpected) experimental confirmations!

atomistic

van der Waals heterostructures

Precedent of the “non-Hermitian skin effect”

PHYSICAL REVIEW B **97**, 121401(R) (2018)

Rapid Communications

Non-Hermitian robust edge states in one dimension: Anomalous localization and eigenspace condensation at exceptional points

V. M. Martinez Alvarez,¹ J. E. Barrios Vargas,² and L. E. F. Foa Torres^{2,*}

¹Departamento de Física, Laboratório de Física Teórica e Computacional, Universidade Federal de Pernambuco, Recife 50670-901, Pernambuco, Brazil

²Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago 837.0415, Chile

(Received 25 August 2017; revised manuscript received 17 February 2018; published 7 March 2018)

Capital to topological insulators, the bulk-boundary correspondence ties a topological invariant computed from the bulk (extended) states with those at the boundary, which are hence robust to disorder. Here we put forward a different ordering unique to non-Hermitian lattices whereby a pristine system becomes devoid of extended states, a property which turns out to be robust to disorder. This is enabled by a peculiar type of non-Hermitian degeneracy where a macroscopic fraction of the states coalesce at a single point with a geometrical multiplicity of 1.

DOI: 10.1103/PhysRevB.97.121401

bilayer graphene
Floquet topological
Dirac point
non-Hermitian
driving frequency
optical phonons
persistent current
topological phases
multilayer graphene
quantum dot coupled
applied voltage
Chern number
level spacing
phonons
quantum dot
lock space
aphene-based
ouble barrier
litage pumped current
antum transport
gapless
interface states
spin pumping
ation
yers

bes
electron system
excitation
edge states
electrodes
Haldane
ry-Perot
ge and spin
phonon-assisted
gain and loss
phase fluctuations
two-terminal
low bias
ion transport

● Recently used
● Not recently used



Víctor Manuel Martínez Álvarez (UFPe Brazil)



José Eduardo Barrios Vargas (U-Chile)

Organization of these lectures

Brief motivation: What's the hype about topological matter?

Topology in one dimension: The SSH model

Graphene's bandstructure

The many faces of the Quantum Hall Effect

From Haldane's model to Topological Insulators

New directions

All the material from these lectures will be here:



Quantum effects are everywhere
but in some materials they become evident
at a wider range of energy and length scales.

Quantum materials include superconductors,
graphene, **topological insulators**, Weyl semimetals, quantum spin liquids,
and more.

Sources of the underlying properties include:
reduced dimensionality and **interactions**.

Quantum Hall effects



Macroscopic quantum phenomena

Quantum Hall effects



artificial systems
under
extreme conditions



materials
and
applications



Quantum Hall effects

PHYSICAL REVIEW LETTERS **122**, 200001 (2019)

Essay: Quantum Hall Effect and the New International System of Units

Klaus von Klitzing

Max Planck Institut für Festkörperforschung, Heisenbergstraße 1, D-70569 Stuttgart, Germany



(Received 18 March 2019; published 20 May 2019)

On 20 May 2019 the International System of Units will switch over to a new, globally accepted set of definitions based on constants of nature. The quantum Hall effect, discovered 39 years ago, plays an essential role in this development and will now contribute to a new definition of the kilogram. Ironically, the title of publication in *Physical Review Letters* that introduced the quantum Hall effect is, in hindsight, inaccurate in view of the revised units [K. von Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, *Phys. Rev. Lett.* **45**, 494 (1980)].

DOI: [10.1103/PhysRevLett.122.200001](https://doi.org/10.1103/PhysRevLett.122.200001)



materials
and
applications





Quantum Hall effects

THE SI

The SI — the modern metric system — has seven base units from which all other measurement units can be derived. On May 20, 2019, four of them — the kilogram, kelvin, ampere and mole — were redefined in terms of constants of nature. The remaining three — the second, meter, and candela — are already based on universal constants.

Click on the SI symbols below for more information.



materials
and
applications



Against all odds, topological properties turned out to be ubiquitous

Quantum Hall effects

[Submitted on 20 May 2021]

arXiv:2105.09954 [cond-mat.mtrl-sci]

All Topological Bands of All Stoichiometric Materials

Maia G. Vergniory,^{1,2,*} Benjamin J. Wieder,^{3,4,5,*} Luis Elcoro,⁶ Stuart S. P. Parkin,⁷ Claudia Felser,⁸ B. Andrei Bernevig,⁵ and Nicolas Regnault^{5,9,†}

38,298 unique materials (nonmagnetic)

52.65% of all materials are topological at EF



materials
and
applications



Organization of these lectures

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Topology in one dimension: The SSH model

Graphene's bandstructure

The many faces of the Quantum Hall Effect

From Haldane's model to Topological Insulators

New directions

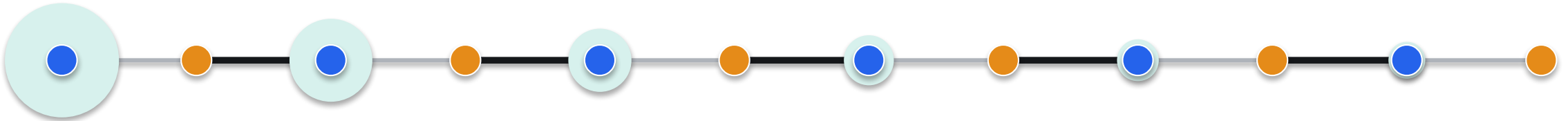
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Lecture 1

Topology in one dimension: the SSH model

From dimerization to protected boundary states



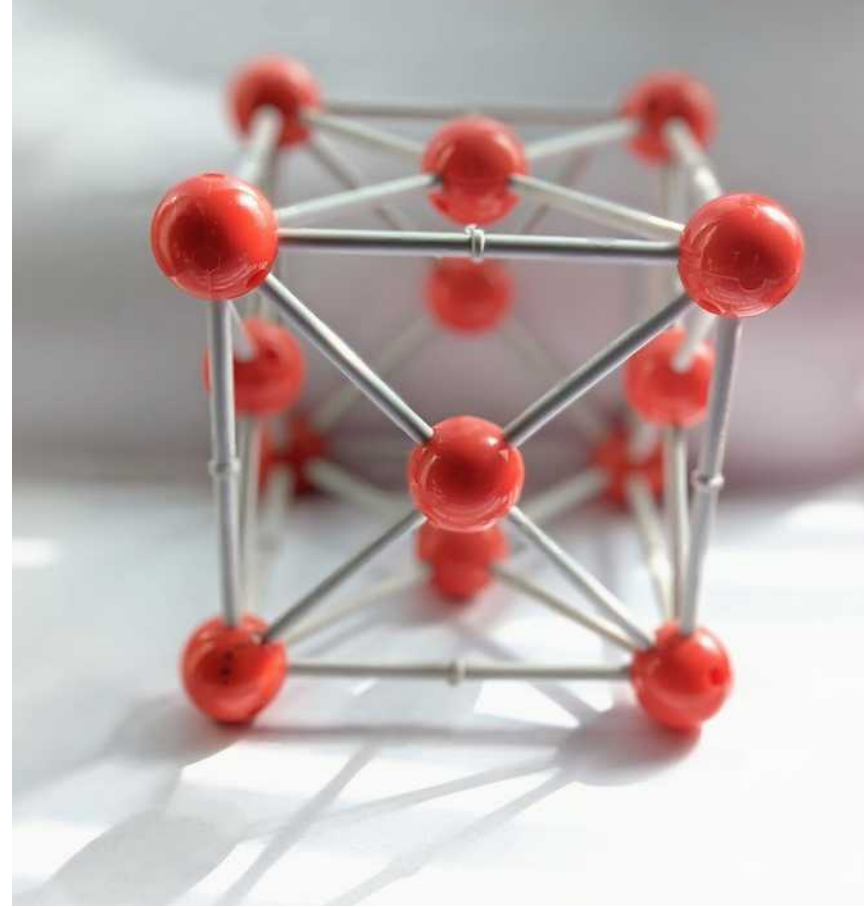
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Historical context for the SSH model



Werner Heisenberg and Felix Bloch around 1931



A long history of electron-phonon interaction in solids

ANNALEN DER PHYSIK

5. F O L G E, 1930, B A N D 4, H E F T 2

Zur Theorie der elektrischen und thermischen Leitfähigkeit von Metallen

Von R. Peierls

(Mit 1 Figur)

Einleitung

Die Frage nach dem *Mechanismus* der Elektronenbewegung in Metallen ist seit den Untersuchungen von Bloch¹⁾ als ziemlich abgeschlossen anzusehen. Es wurde dort u. a. gezeigt, daß sich ein Elektron in einem rein periodischen Kraftfeld ungehindert bewegen kann, daß also ein Widerstand immer erst durch die thermische Bewegung des Gitters zustande kommt. Die Verhältnisse liegen ähnlich wie bei der Streuung von Röntgenstrahlen; Houston²⁾ und Frenkel³⁾ versuchten, die Wechselwirkung zwischen den Elektronen und den Gitterschwingungen genau analog zu der bekannten Beugungstheorie für Röntgenstrahlen zu behandeln. Bloch zeigte jedoch, daß

The Electrical Resistance of Metals.

P. W. Bridgman

Phys. Rev. **17**, 161 – Published 1 February 1921

REVIEWS OF MODERN PHYSICS

VOLUME 23, NUMBER 3

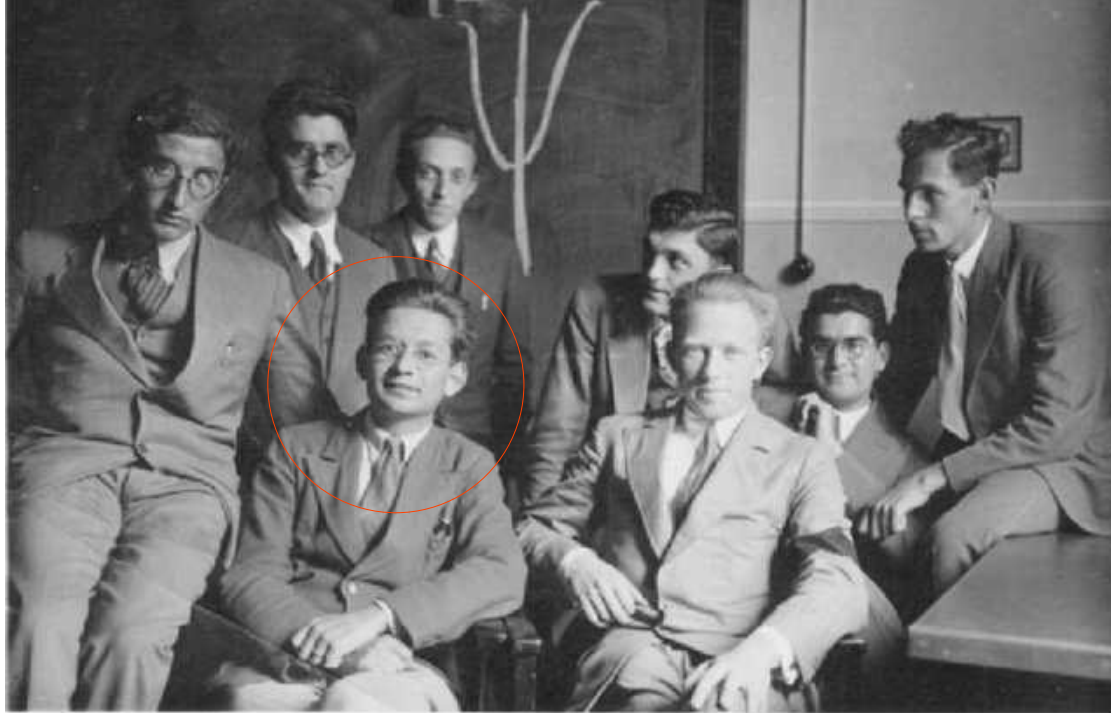
JULY, 1951

Electron-Vibration Interactions and Superconductivity

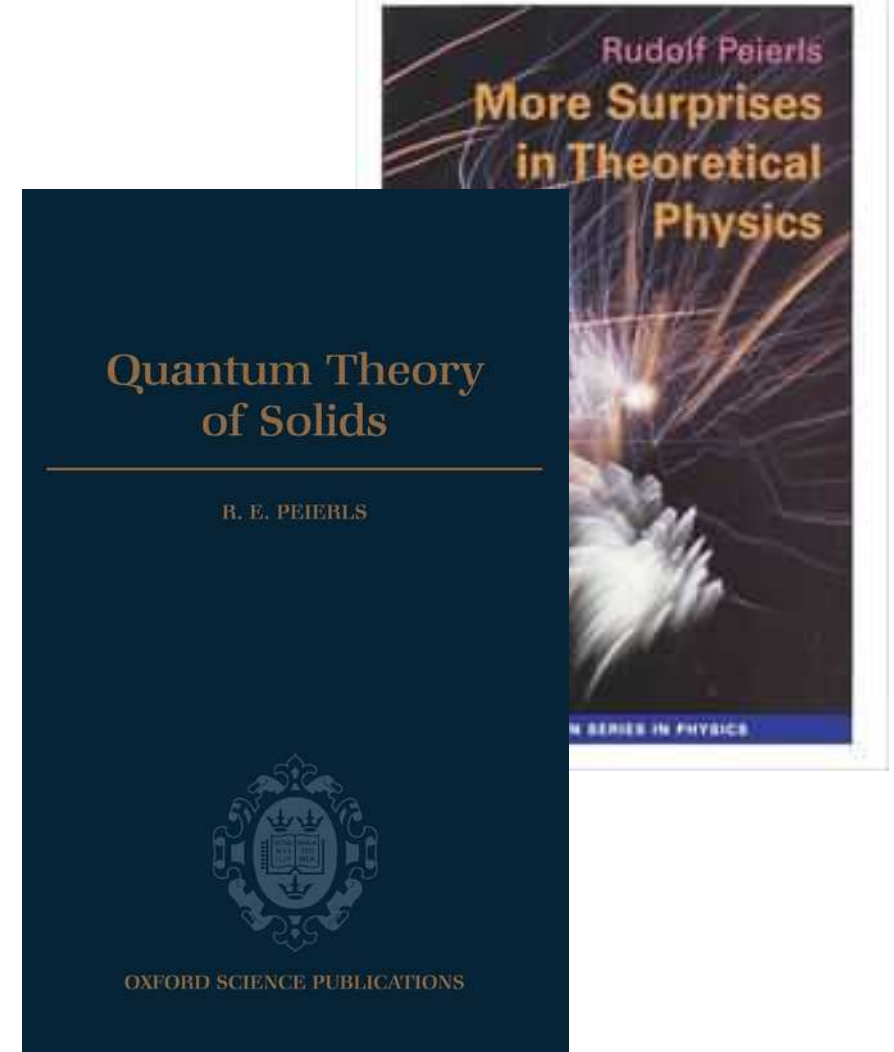
J. BARDEEN*

Bell Telephone Laboratories, Murray Hill, New Jersey

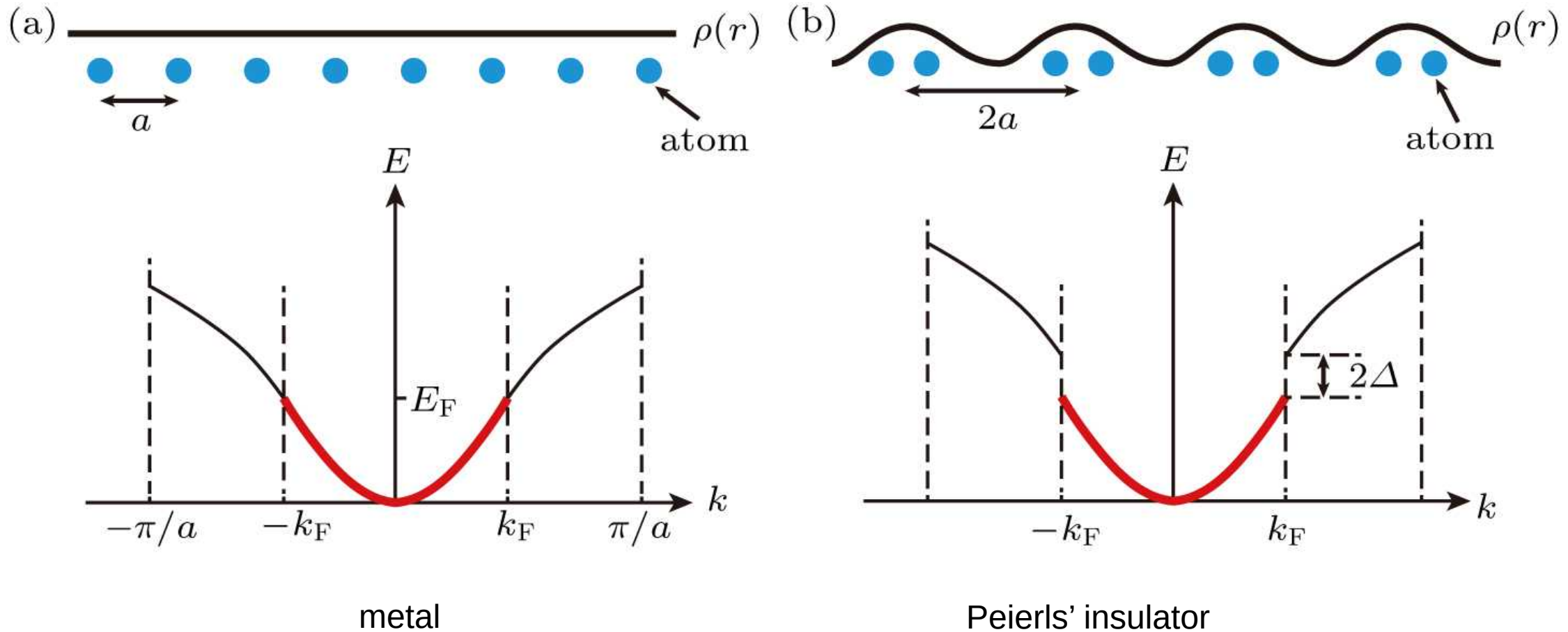
A long history of e-ph interaction in solids: The Peierls' transition



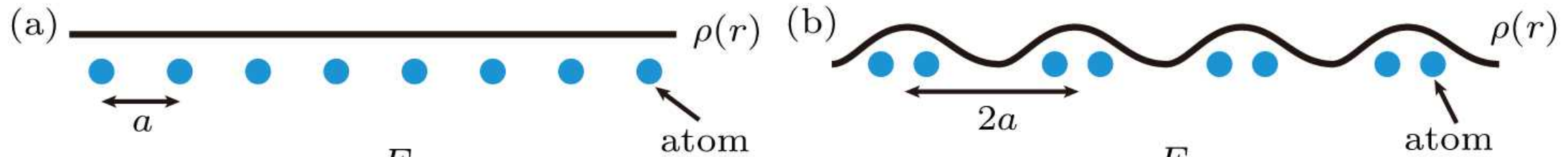
Heisenberg's Institute in Leipzig, 1931. Front row (L-R): George Placzek (sitting on desk), **Rudolf Peierls** and Werner Heisenberg; back row (L-R): Giovanni Gentile Jr., Gian Carlo Wick, Felix Bloch, Viktor Weisskopf and Fritz Sauter.
Copyright: Alessandra Gentile



A long history of e-ph interaction in solids: The Peierls' transition



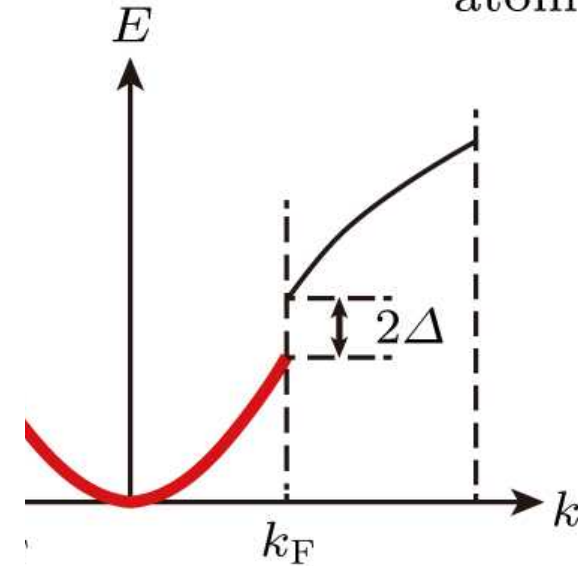
A long history of e-ph interaction in solids: The Peierls' transition



placement δ . The behavior of the reduction in electronic energy for small displacement is therefore as

$$- \delta^2 \log \delta. \quad (2.3.5)$$

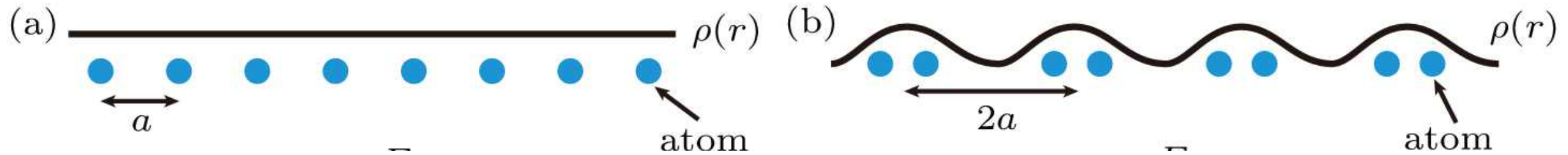
This is interesting because there may be other effects favoring the regular spacing, $\delta = 0$, such as the repulsion between the atomic cores, but these will have an energy varying as δ^2 , and therefore the electron energy must dominate for small displacement. This suggests that the periodic chain must always be unstable.



metal

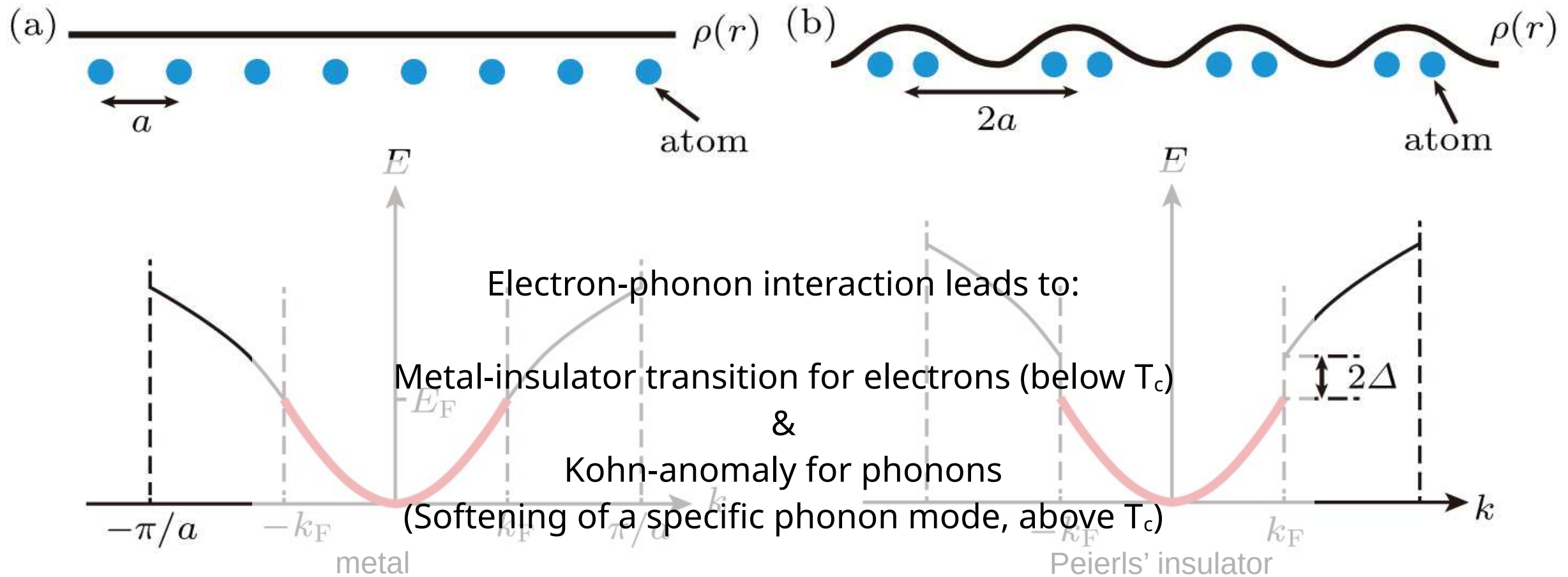
Peierls' insulator

A long history of e-ph interaction in solids: The Peierls' transition



Behavior as a function of temperature, CDWs, critical temperature T_c

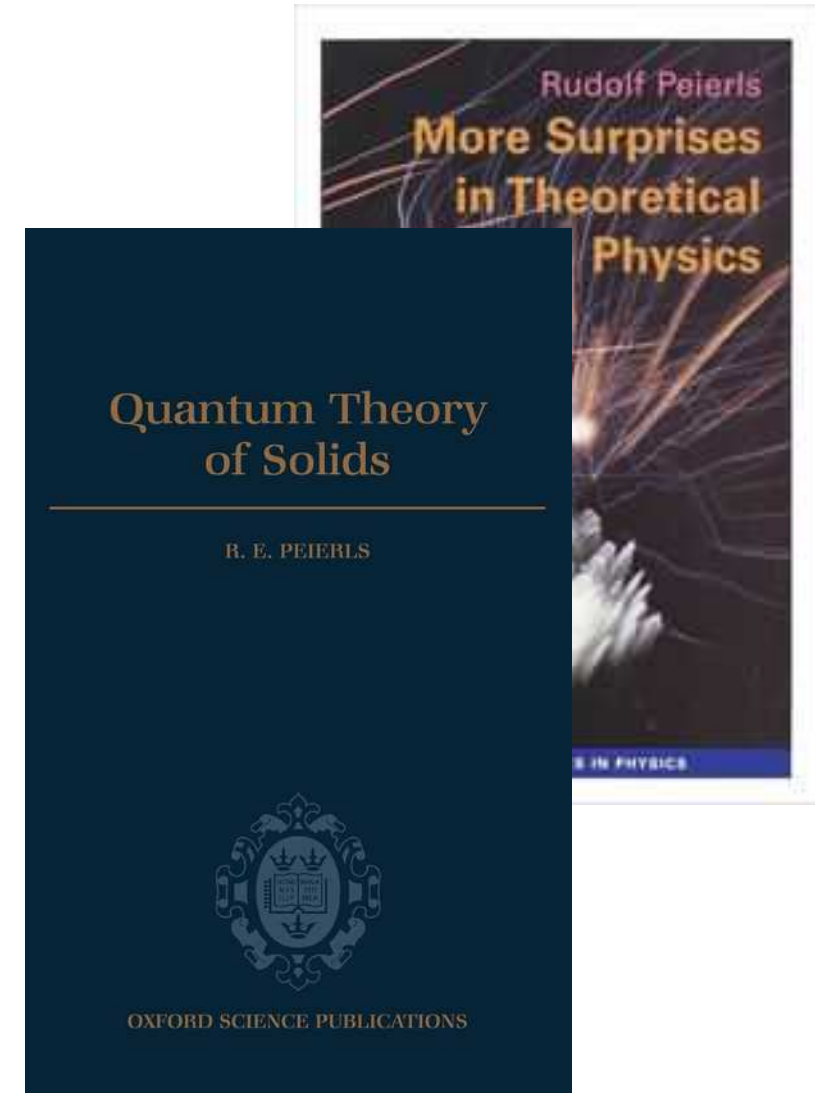
A long history of e-ph interaction in solids: The Peierls' transition



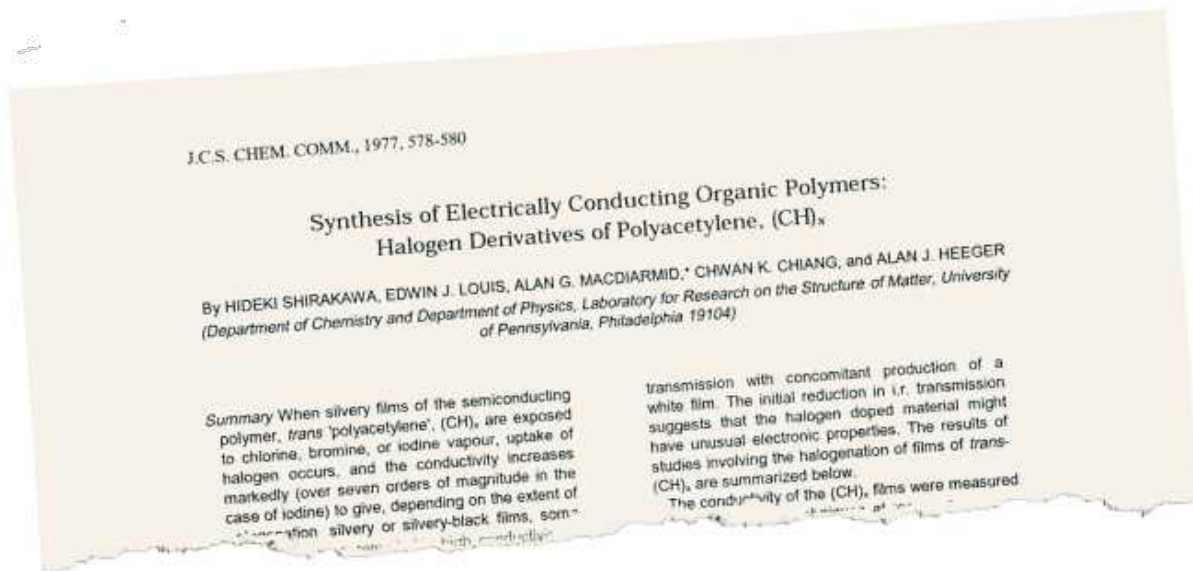
A long history of e-ph interaction in solids: The Peierls' transition

This instability came to me as a complete surprise when I was tidying material for my book (Peierls 1955), and it took me a considerable time to convince myself that the argument was sound. It seemed of only academic significance, however, since there are no strictly one-dimensional systems in nature (and if there were, they would become disordered at any finite temperature; see *Surprises*, sec. 4.1). I therefore did not think it worth publishing the argument, beyond a brief remark in the book, which did not even mention the logarithmic behavior.

In this case the first surprise was in the mathematical result for the one-dimensional case; a second surprise was that this had some connection with reality.



A long history of e-ph interaction in solids: Conducting polymers



Hideki Shirakawa, Alan MacDiarmid and Alan Heeger, Nobel prize in Chemistry 2000

A long history of e-ph interaction in solids: Conducting polymers

VOLUME 42, NUMBER 25

PHYSICAL REVIEW LETTERS

18 JUNE 1979

Solitons in Polyacetylene

W. P. Su, J. R. Schrieffer, and A. J. Heeger

Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104

(Received 15 March 1979)

We present a theoretical study of soliton formation in long-chain polyenes, including the energy of formation, length, mass, and activation energy for motion. The results provide an explanation of the mobile neutral defect observed in undoped $(\text{CH})_x$. Since the soliton formation energy is less than that needed to create band excitation, solitons play a fundamental role in the charge-transfer doping mechanism.

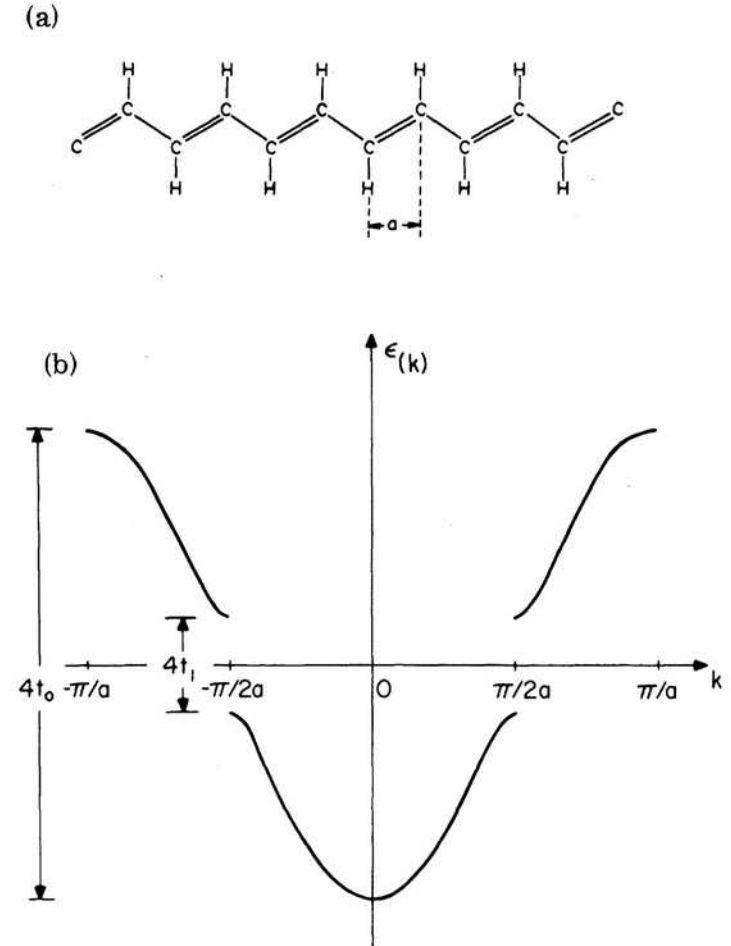
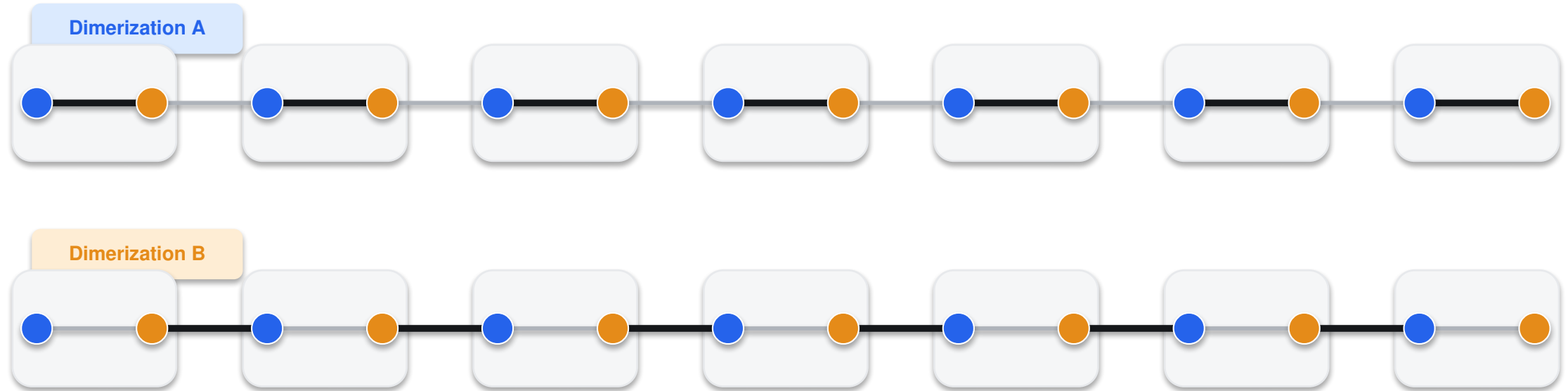


FIG. 1. (a) *Trans* configuration of $(\text{CH})_x$. $a=1.2 \text{ \AA}$.
(b) π -band structure of perfectly dimerized $(\text{CH})_x$.

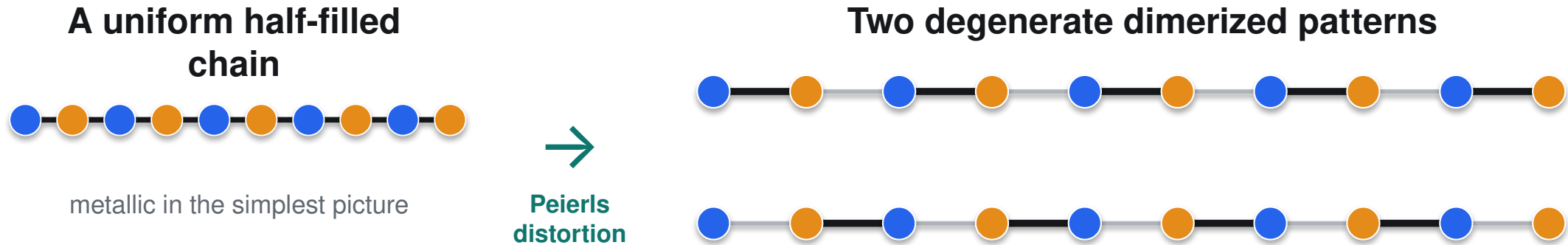
Two gapped chains. The same insulator?



Both have **two sites per cell** · two bands · a finite bulk gap

Can a bulk property distinguish them?

Where the model comes from: polyacetylene



A lattice distortion opens a gap — but it also creates two possible insulating ground states.

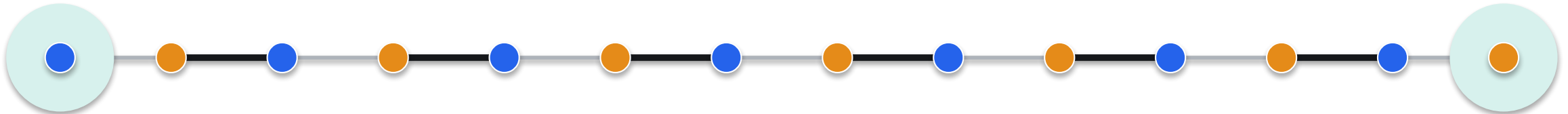
W. P. Su, J. R. Schrieffer & A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979)

The distinction becomes visible at a boundary

Strong bonds at the ends



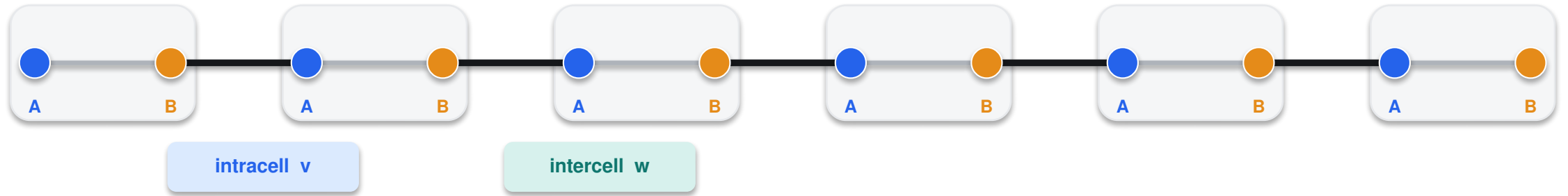
Weak bonds at the ends



unpaired end degrees of freedom

We now look for a bulk statement that predicts this boundary difference.

The SSH Hamiltonian



$$\hat{H} = \sum_n \left(v a_n^\dagger b_n + w a_{n+1}^\dagger b_n + \text{h.c.} \right)$$

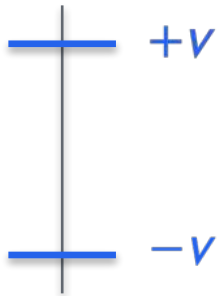
- one orbital per site; two sites A and B per unit cell
- spin is suppressed for simplicity
- nearest-neighbor hopping only; take both hoppings to be positive

Two limits you can solve by eye

$w = 0$ only intracell bonds

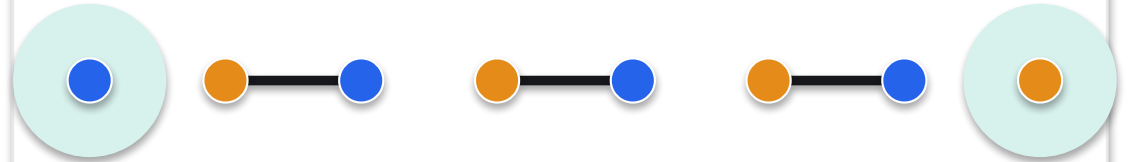


isolated dimers — every site is paired

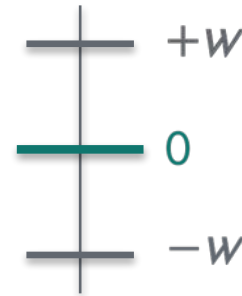


two flat bands at $\pm v$
nothing inside the gap

$v = 0$ only intercell bonds



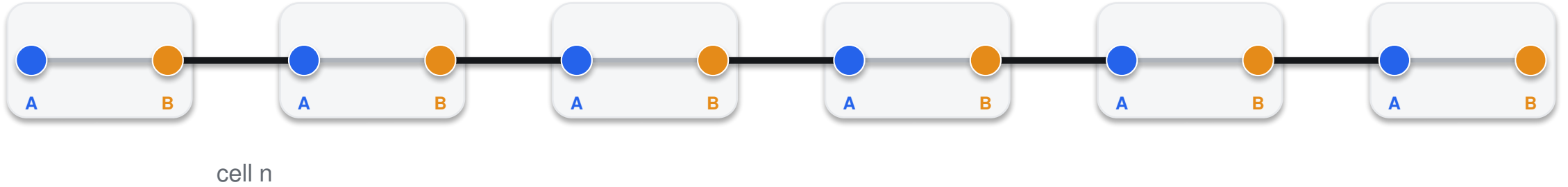
the two end sites decouple completely



two states pinned exactly
at zero energy — one per end

Away from these limits the bonds compete. Which features survive at general v, w — and why?

From the lattice to momentum space



$$a_n = \frac{1}{\sqrt{N}} \sum_k e^{ikna} a_k$$

$$b_n = \frac{1}{\sqrt{N}} \sum_k e^{ikna} b_k$$

a = unit-cell length · N = number of cells

$$|\psi_k\rangle = (a_k \quad b_k)^T$$

The bulk problem is now a 2×2 eigenvalue equation.

A two-band Bloch Hamiltonian

$$H(k)|u_k\rangle = E(k)|u_k\rangle$$

$$H(k) = \begin{pmatrix} 0 & q^*(k) \\ q(k) & 0 \end{pmatrix}$$

$$q(k) = v + we^{ika}$$

all the geometry is
contained in one complex number

The diagonal vanishes because there are no A–A or B–B terms.

The Hamiltonian as a vector

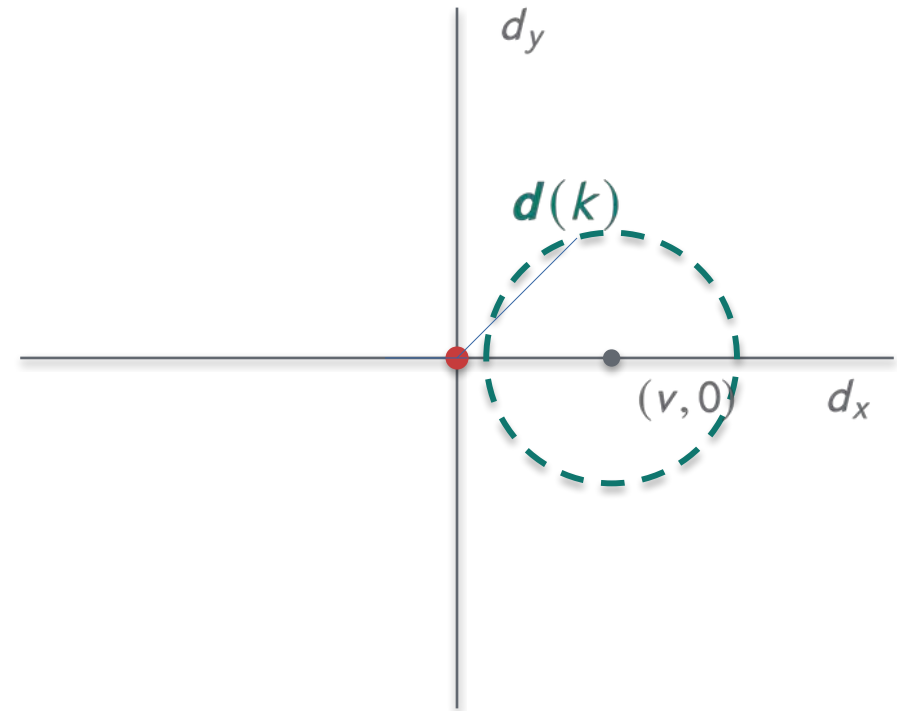
$$H(k) = d_x(k)\sigma_x + d_y(k)\sigma_y$$

$$d_x(k) = v + w \cos(ka)$$

$$d_y(k) = w \sin(ka)$$

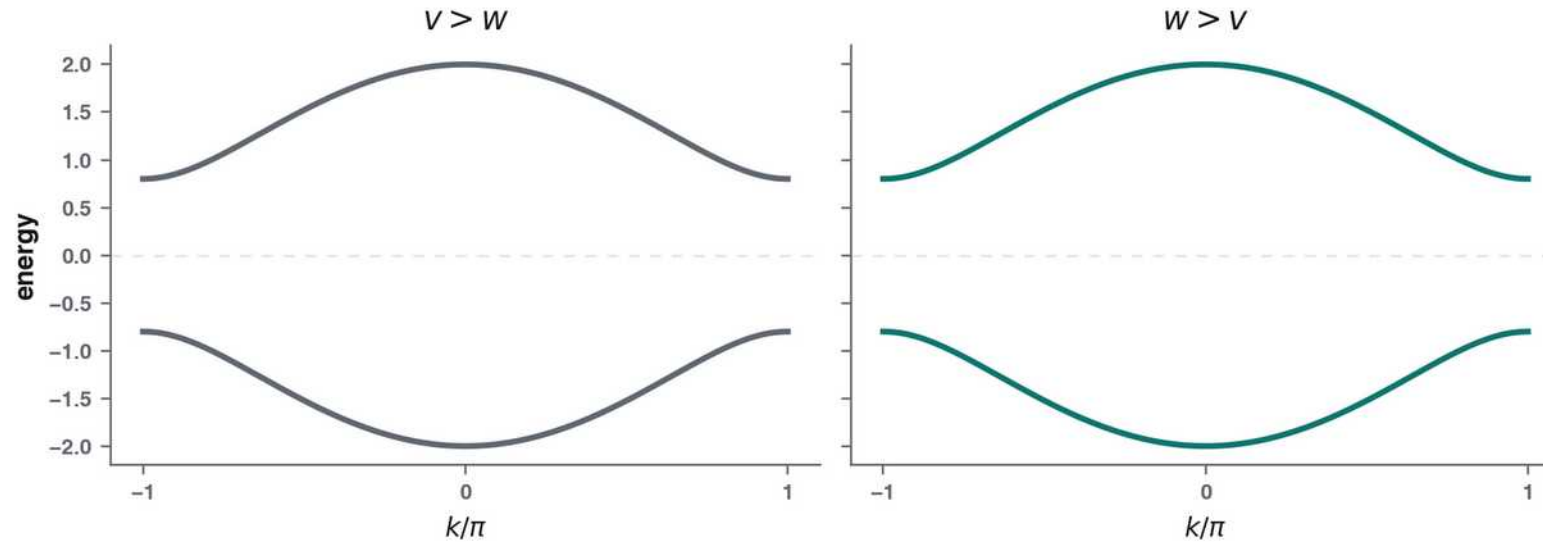
$$d_z(k) = 0$$

As momentum changes, the Hamiltonian vector draws a closed path in a plane.



the tip traces the dashed circle once per Brillouin zone

The bands do not yet reveal the distinction



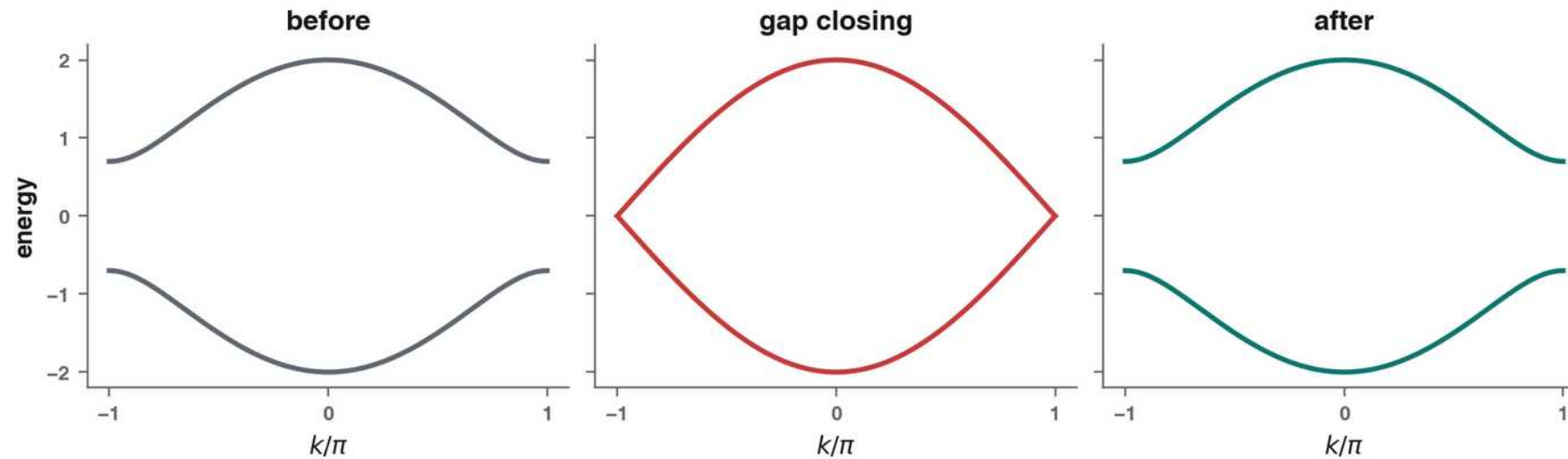
$$E_{\pm}(k) = \pm|q(k)|$$

Both sides have

- two bands
- a symmetric spectrum
- a finite gap

Something besides the eigenvalues is needed.

To move between the two regimes, the gap must close



$$V = W$$

$$q(\pi/a) = 0$$

$$E_{\pm}(\pi/a) = 0$$

The insulating state is undefined at the transition.

Whatever bulk number distinguishes the two regimes, it can change only at a gap closing.

The protecting constraint: chiral symmetry



hopping connects A only to B

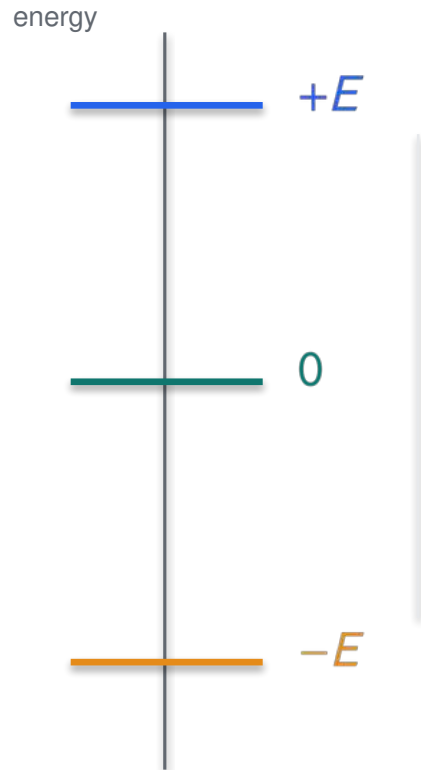
$$\Gamma = \sigma_z$$

$$\Gamma H(k) \Gamma^{-1} = -H(k)$$

Equivalently: $\{\Gamma, H(k)\} = 0$

Chiral symmetry forbids an out-of-plane component: the path of $d(k)$ is pinned to the plane.

What chiral symmetry buys us



Spectral pairing

$$H|\psi\rangle = E|\psi\rangle$$

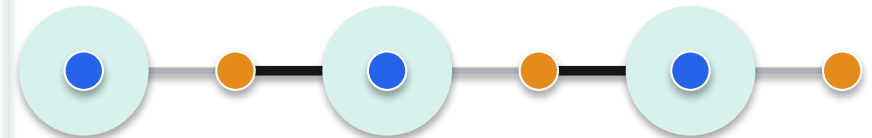
$$H\Gamma|\psi\rangle = -E\Gamma|\psi\rangle$$

Where eigenstates live

$E \neq 0$ — equal weight on A and B

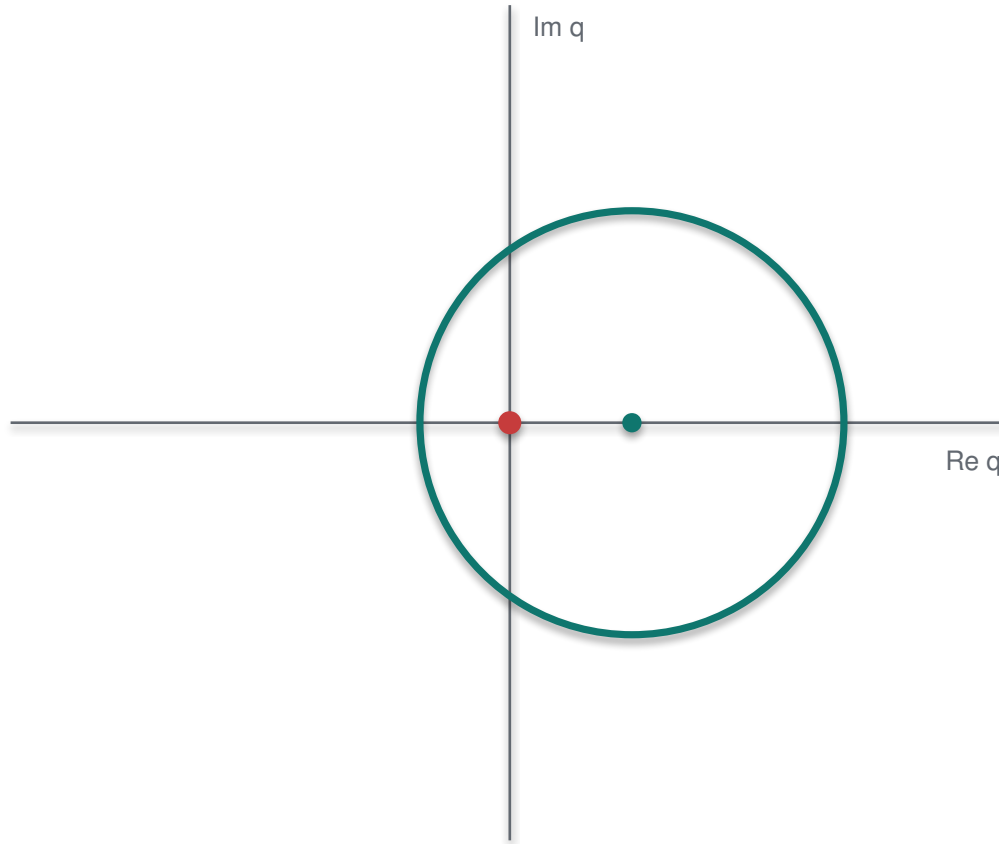


$E = 0$ — one sublattice only



an isolated zero mode can be chosen as a chiral eigenstate — pinned to one sublattice

The Brillouin zone becomes a closed curve



$$q(k) = v + w e^{i k a}$$

$$q(k) = d_x(k) + i d_y(k)$$

the same closed path as the d-vector, read as one complex number

- center: v
- radius: w
- one circuit as k crosses the BZ

The origin is the gap-closing point.

Counting how many times the curve winds

$$q(k) = |q(k)|e^{i\theta(k)}$$

$$\nu = \frac{1}{2\pi} \int_{\text{BZ}} dk \partial_k \theta(k)$$

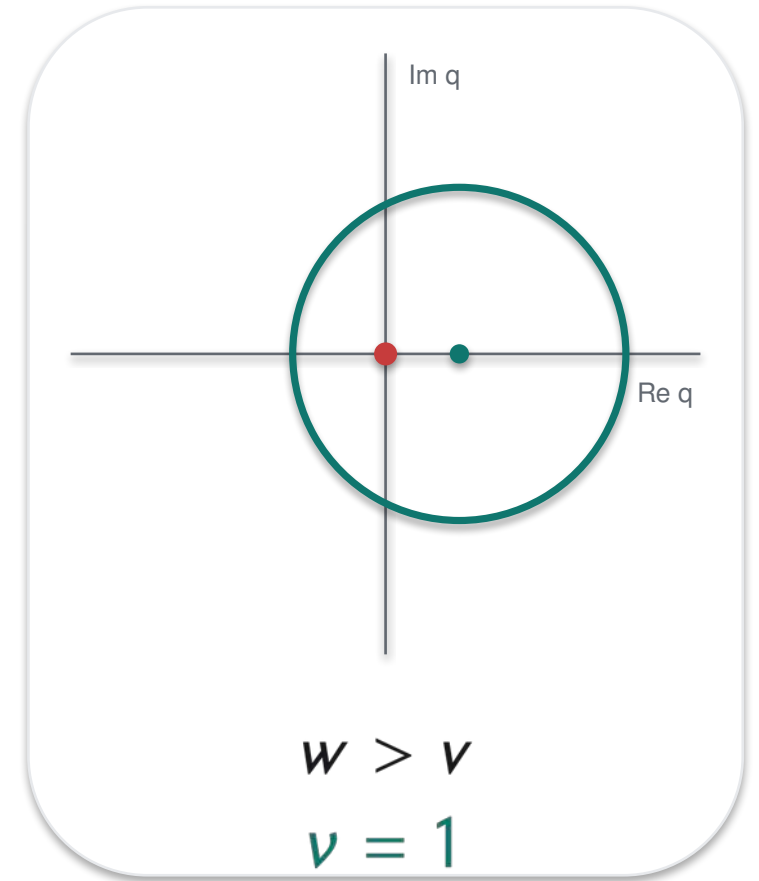
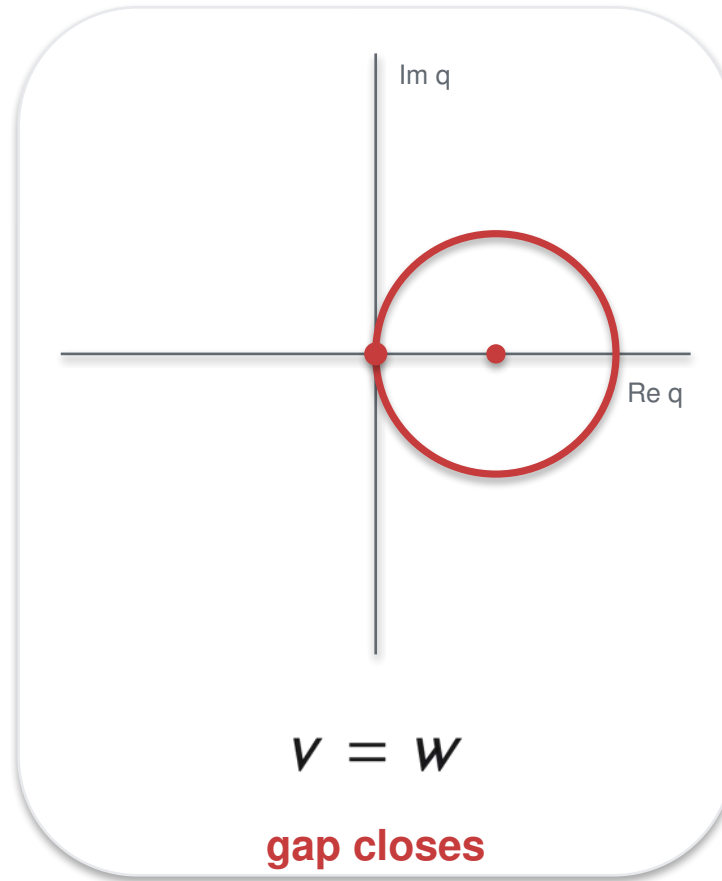
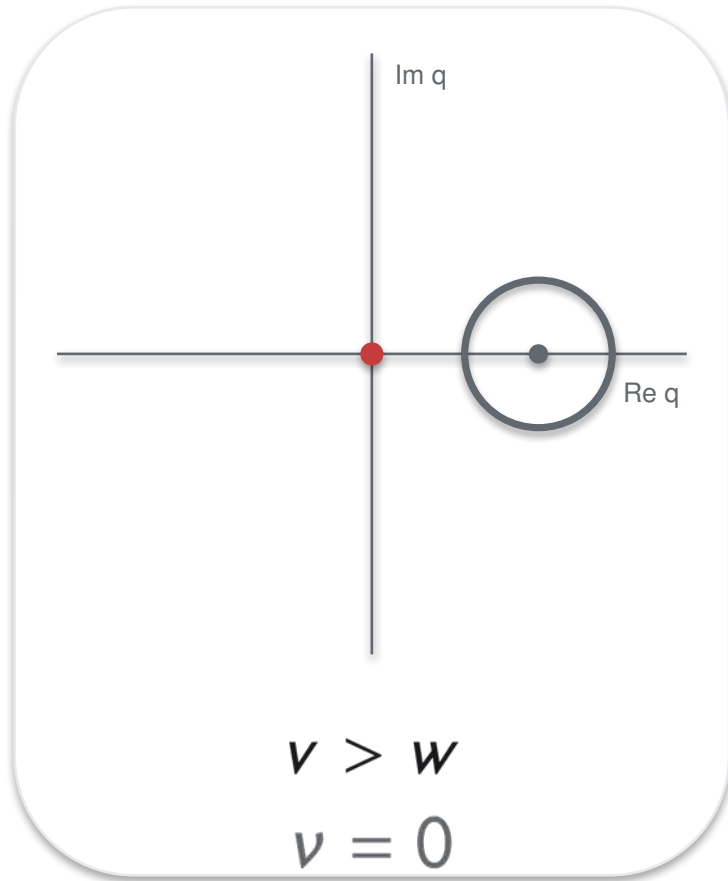
$\nu = 0$ origin outside

$\nu = 1$ origin enclosed

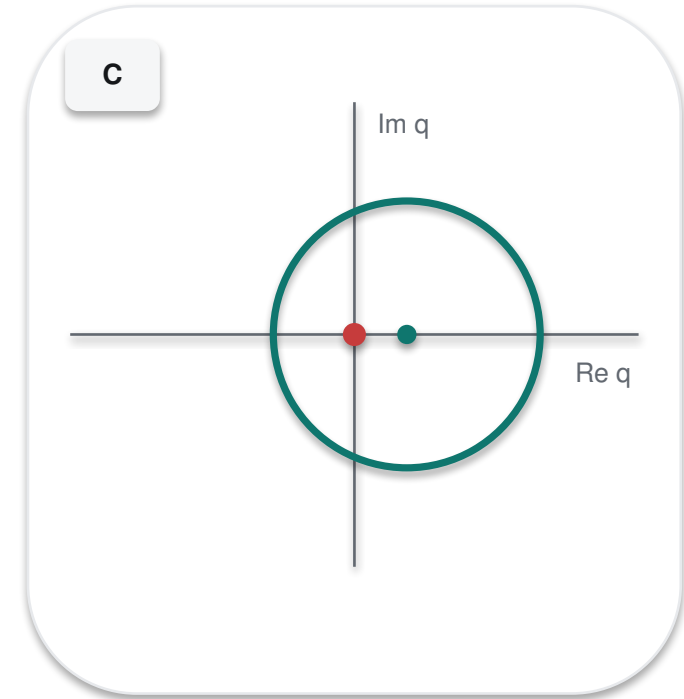
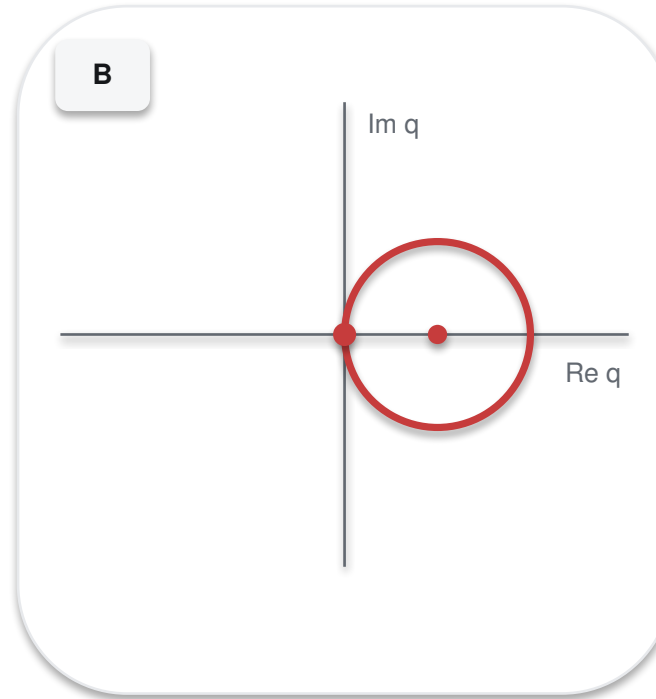
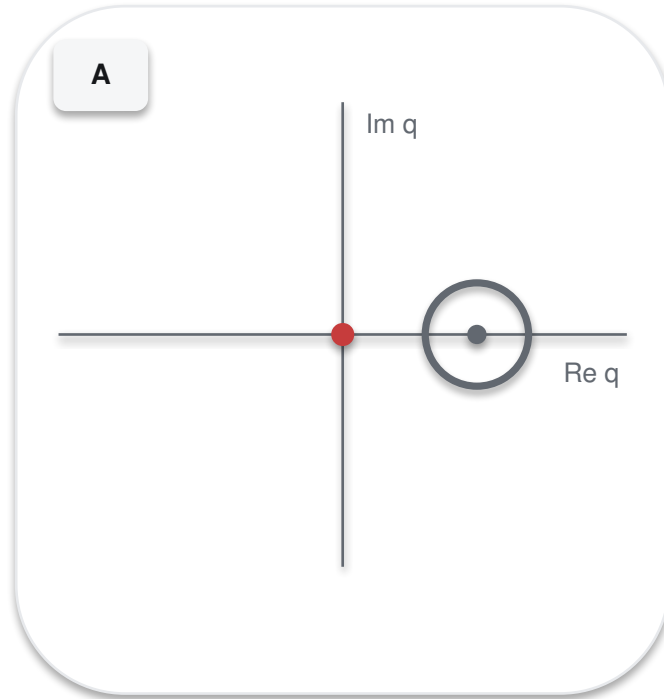
The winding is integer-valued: it cannot drift continuously under a smooth deformation.

J. K. Asbóth, L. Oroszlány & A. Pályi, A Short Course on Topological Insulators, Lect. Notes Phys. 919 (Springer, 2016) · arXiv:1509.02295

Two phases and the transition between them



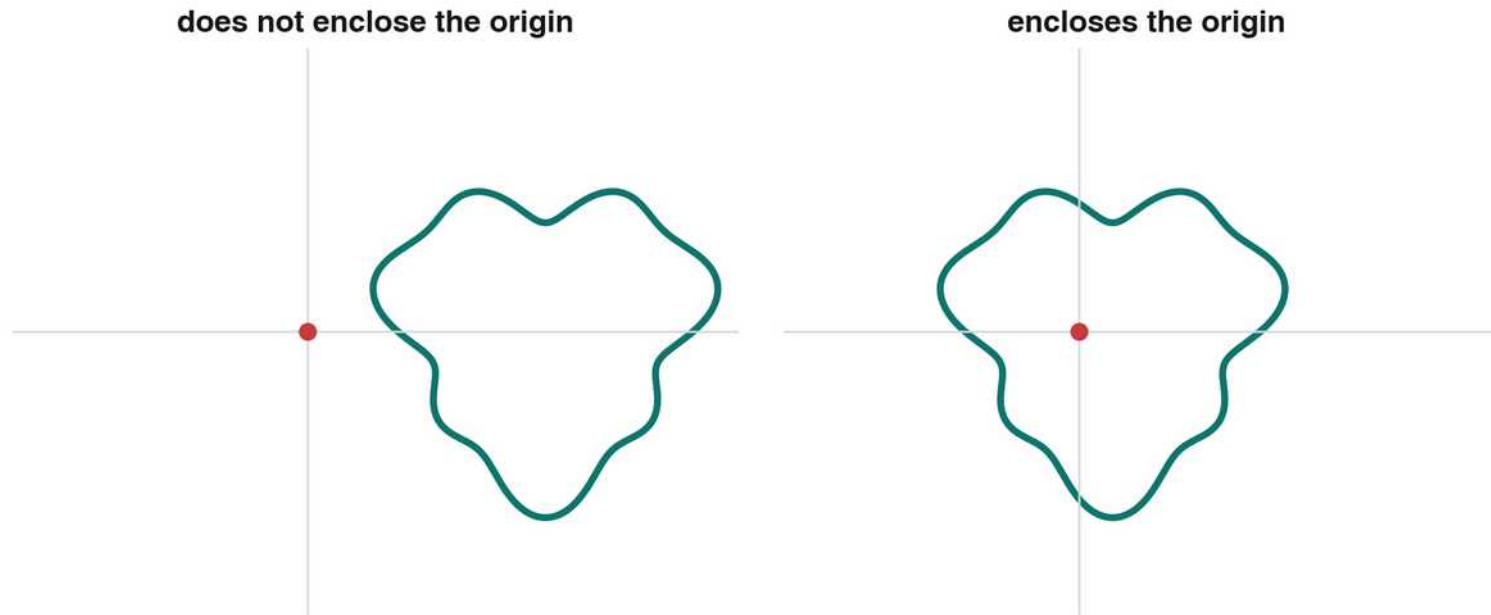
Checkpoint: diagnose the phase without diagonalizing H



For each path: Is the system gapped? What is the winding number?

What must happen before A can become C?

Topology ignores shape, but not the gap



Allowed deformations

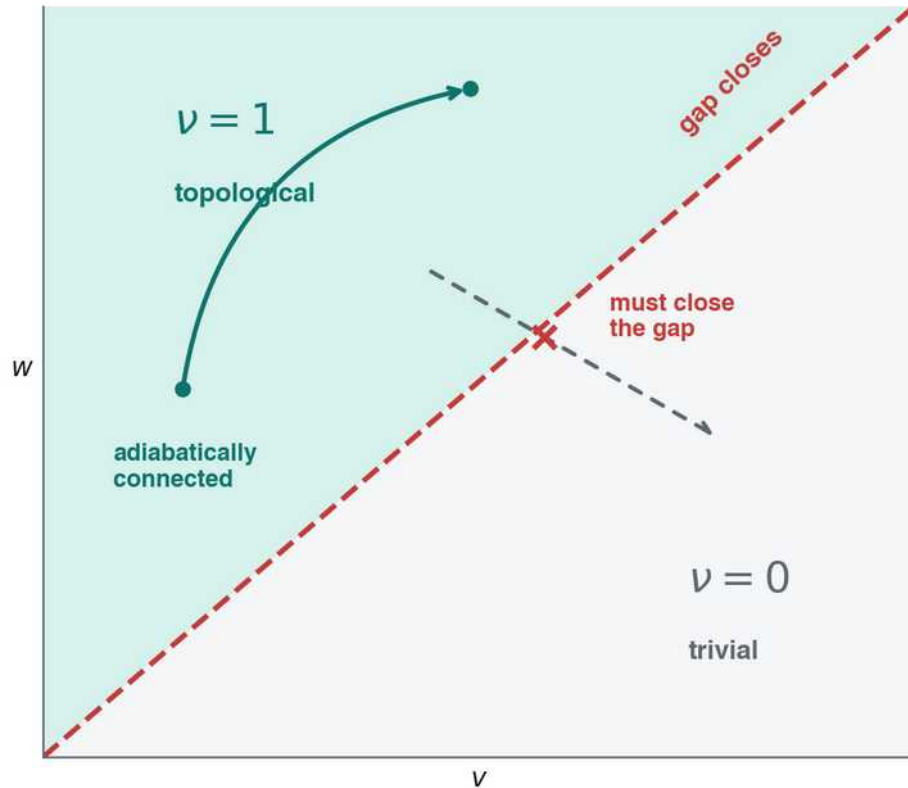
- change v and w smoothly
- distort the loop
- add longer-range A–B bonds

Forbidden shortcut

cross the origin while
remaining an insulator

Different winding numbers label disconnected sets of gapped Hamiltonians.

The phase diagram: two islands that cannot touch

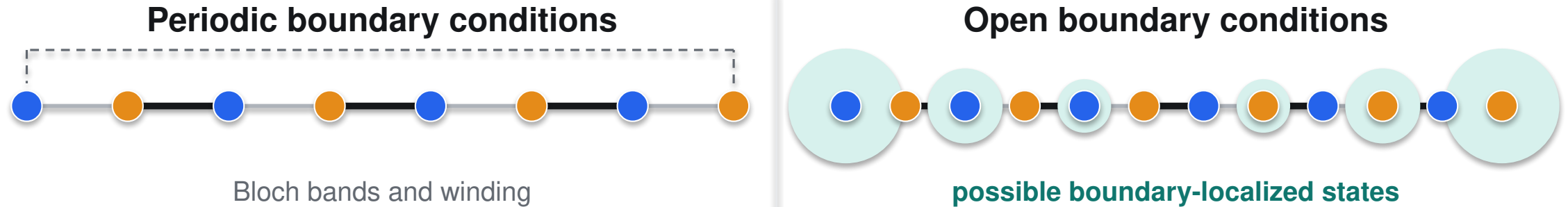


What makes ν a topological invariant?

- it is an integer computed from the gapped bulk
- it cannot change under deformations that keep the gap open and respect chiral symmetry
- so it can only change where the gap closes: the line $v = w$

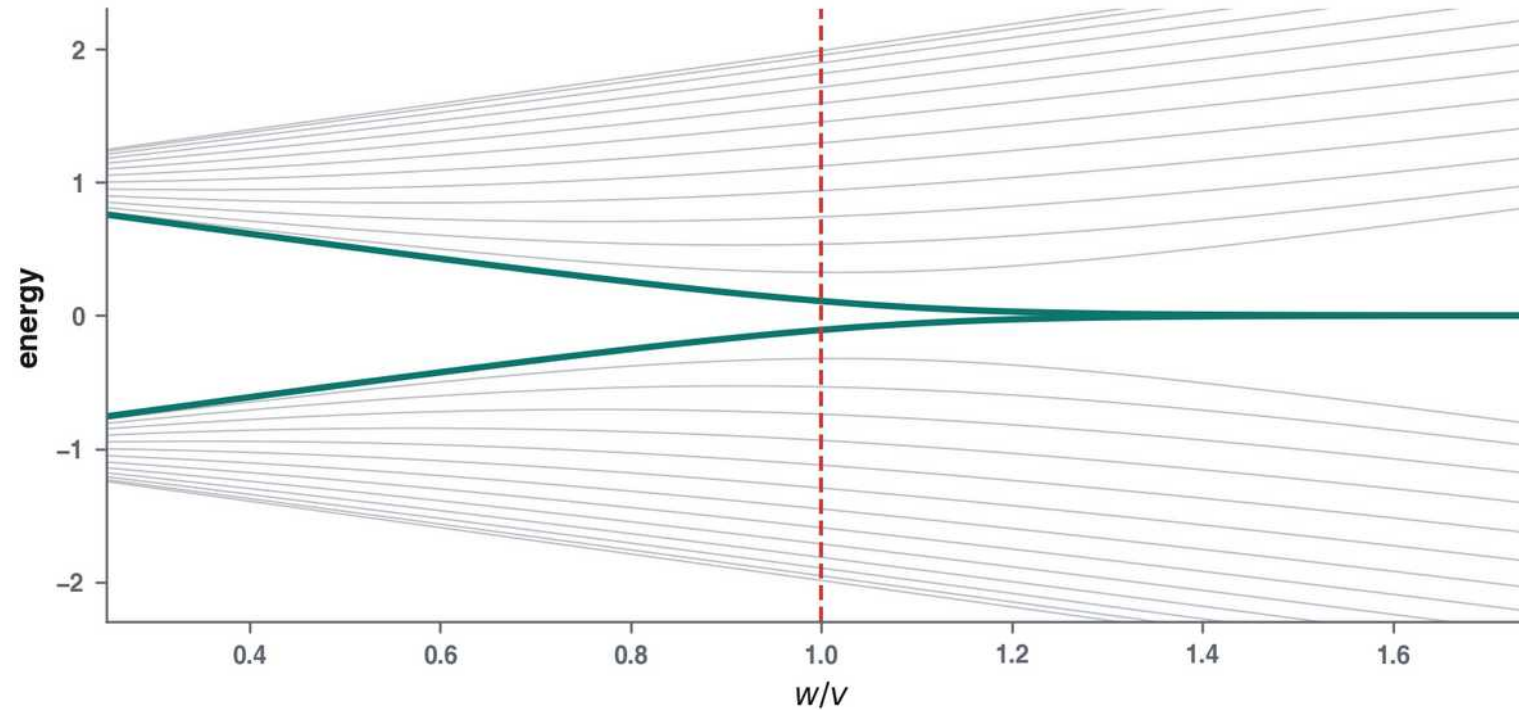
Any two chains inside the same region are adiabatically connected: they are one phase.

The bulk invariant meets a physical boundary



Bulk–boundary correspondence asks whether the bulk winding predicts the boundary spectrum.

The open chain develops in-gap states

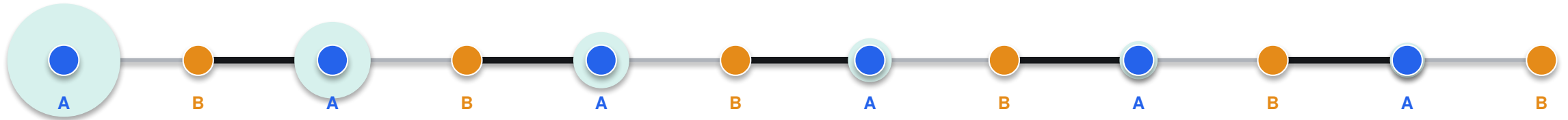


In the nontrivial regime

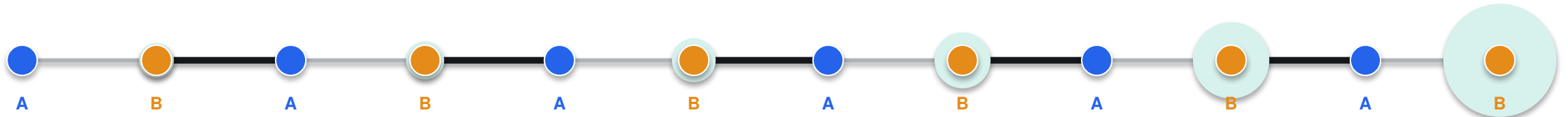
- the bulk remains gapped
- two states approach zero energy
- here $N = 14$ cells — the pair hugs zero as w/v grows

The new states have no periodic-bulk counterpart.

The in-gap states live at the ends



left-edge mode: amplitude only on the A sublattice



right-edge mode: amplitude only on the B sublattice

Localization and sublattice polarization follow directly from zero energy and chiral symmetry.

Deriving the left-edge state



open end

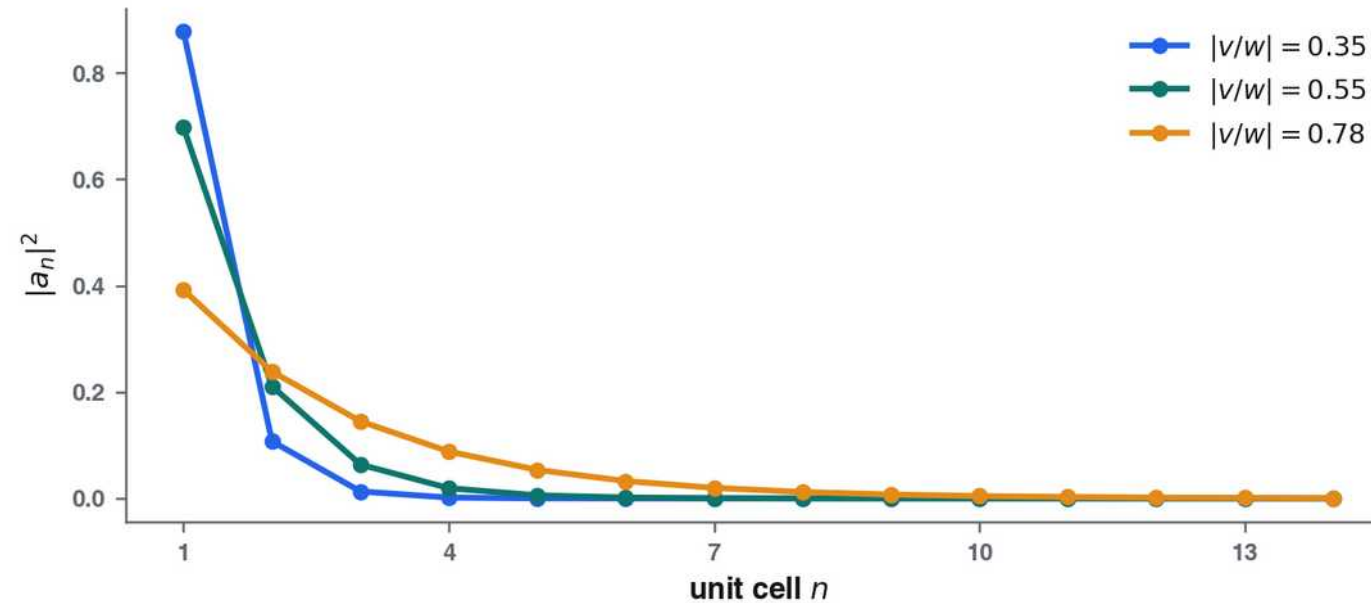
At zero energy, choose

$$b_n = 0$$

$$v a_n + w a_{n+1} = 0 \implies a_{n+1} = -\frac{v}{w} a_n$$

Each unit cell multiplies the amplitude by the negative hopping ratio.

Normalizability selects the topological regime



$$a_n \propto \left(-\frac{v}{w}\right)^{n-1}$$

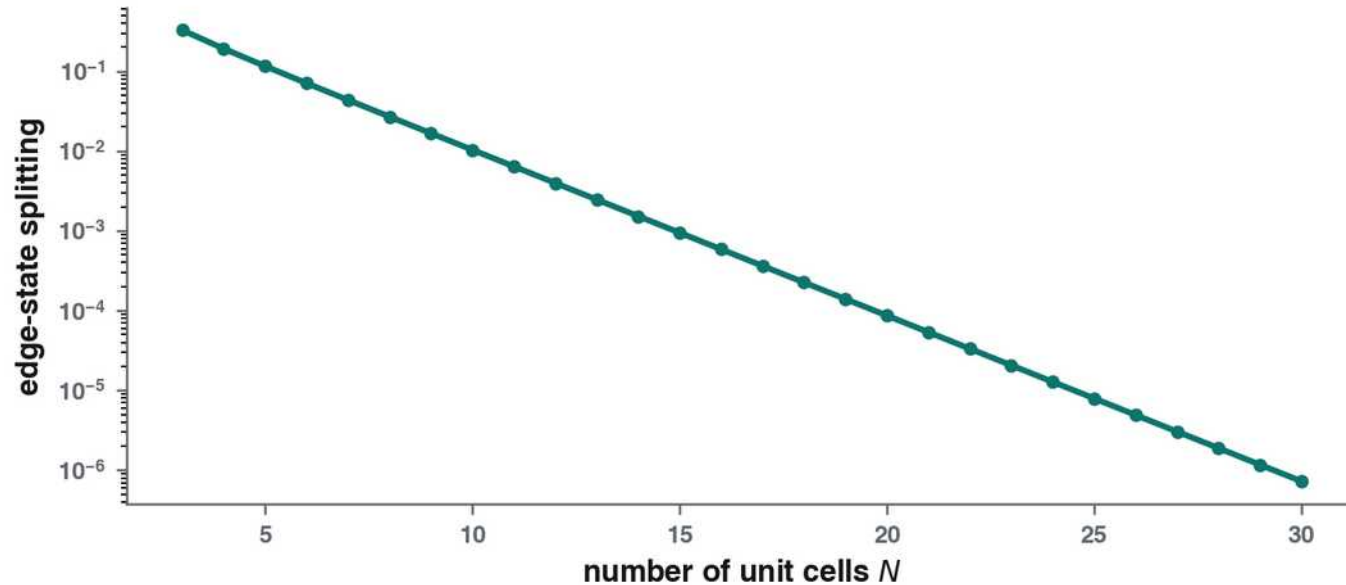
$$\left|\frac{v}{w}\right| < 1$$

normalizable $\iff w > v$

$$\xi^{-1} = \ln \left| \frac{w}{v} \right|$$

The edge-state condition is exactly the nontrivial bulk winding condition.

Finite chains: the two ends can talk to each other

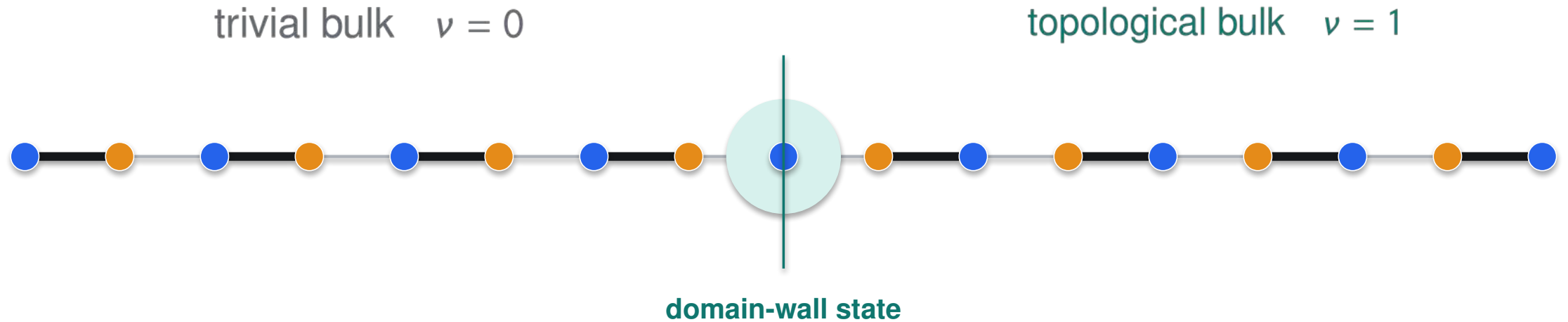


$$\Delta E \sim e^{-L/\xi}$$

- two near-zero levels at opposite energies
- the splitting vanishes exponentially with length

Exact zero energy is recovered in the semi-infinite limit.

An interface makes bulk-boundary correspondence explicit



$$N_{\text{interface}} = |\nu_{\text{right}} - \nu_{\text{left}}|$$

In polyacetylene, this is the soliton associated with a change of dimerization.

The boundary state does not require a perfect lattice



random hopping — still only between A and B

What is preserved?

- chiral symmetry
- a bulk or mobility gap
- the interface winding mismatch

What can remove the state?

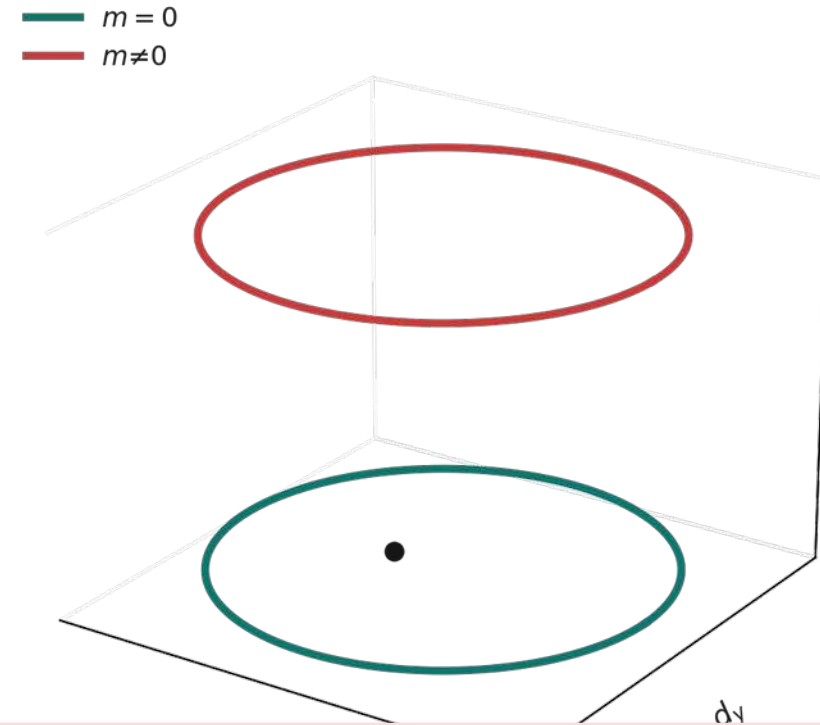
- closing the relevant bulk gap
- breaking the protecting symmetry
- annihilating it with another mode

Without chiral symmetry, the loop can escape

Add a staggered onsite potential

$$H(k) \longrightarrow H(k) + m\sigma_z$$

- the chiral anticommutator no longer vanishes
- a nonzero out-of-plane component is now allowed
- the path can leave the plane



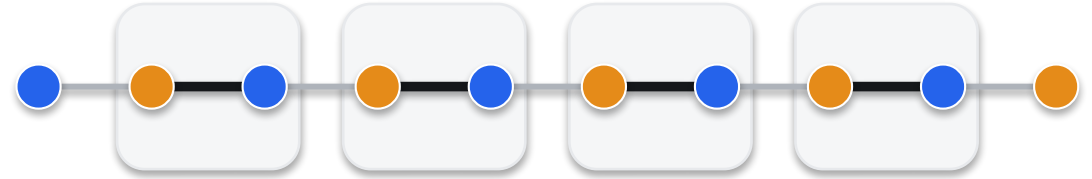
Once the planar constraint is removed, the loop can be contracted without reaching the origin.

A subtle point: unit cell and termination move together

Choice 1: group A and B in the same cell



Choice 2: shift the cell boundary by one site



A relabeling may shift the numerical winding — but it also shifts where the chain is cut.

- compare two bulks using one consistent convention
- the invariant content is the difference across a physical interface
- polarization is defined modulo one electron charge

No unit-cell convention creates a physical edge state by itself.

The SSH diagnostic card

BULK HAMILTONIAN

$$H(k) = d_x \sigma_x + d_y \sigma_y$$

PROTECTING CONSTRAINT

chiral symmetry $\{\Gamma, H\} = 0$

BULK INVARIANT

winding number $\nu \in \mathbb{Z}$

PHASE TRANSITION

the bulk gap closes when the hoppings become equal

BOUNDARY PREDICTION

zero-energy end or interface mode

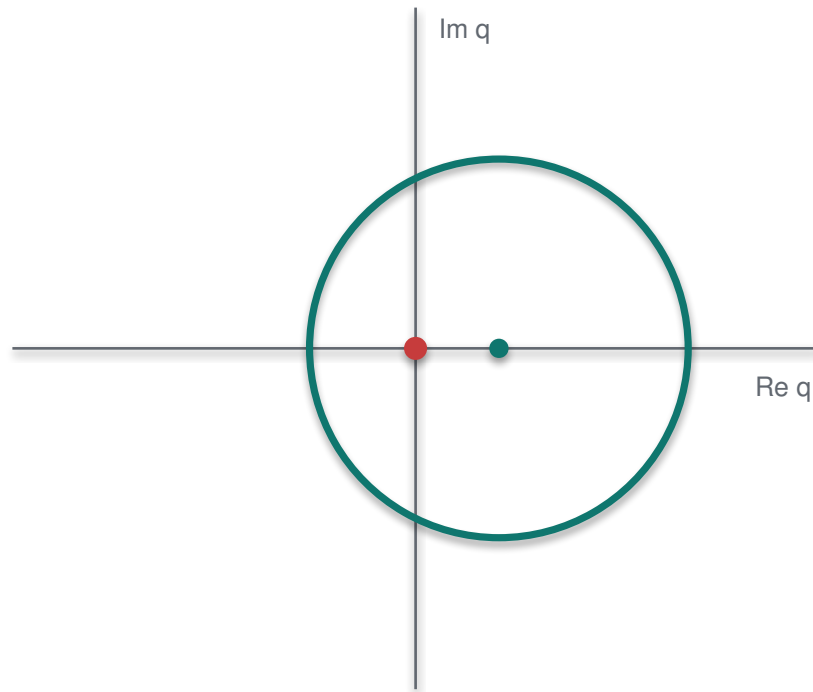
PHYSICAL LANGUAGE

polarization · end charge · soliton

Topology is an obstruction to connecting gapped Hamiltonians while respecting the symmetry.

Further reading: J. K. Asbóth, L. Oroszlány & A. Pályi, *A Short Course on Topological Insulators, Lect. Notes Phys. 919 (Springer, 2016)* · [arXiv:1509.02295](https://arxiv.org/abs/1509.02295)

From a one-dimensional winding to a two-dimensional response



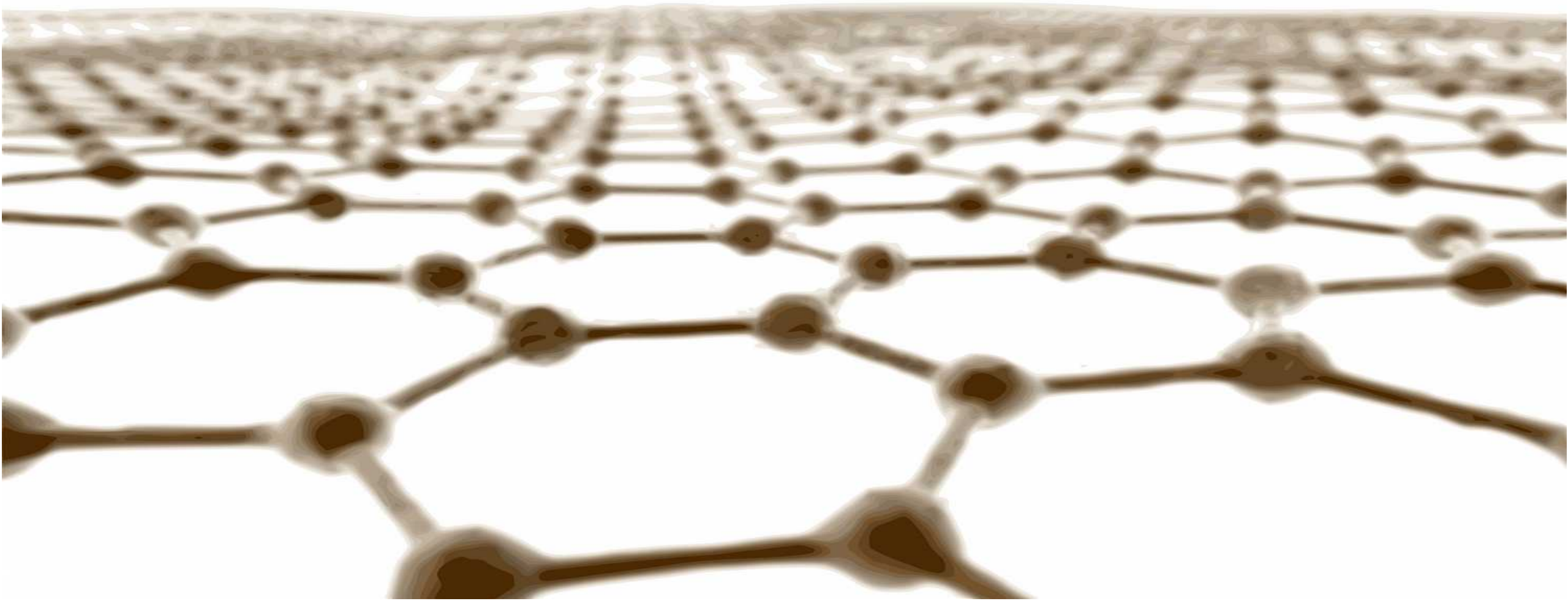
1D BZ → winding



2D bulk → conducting edge?

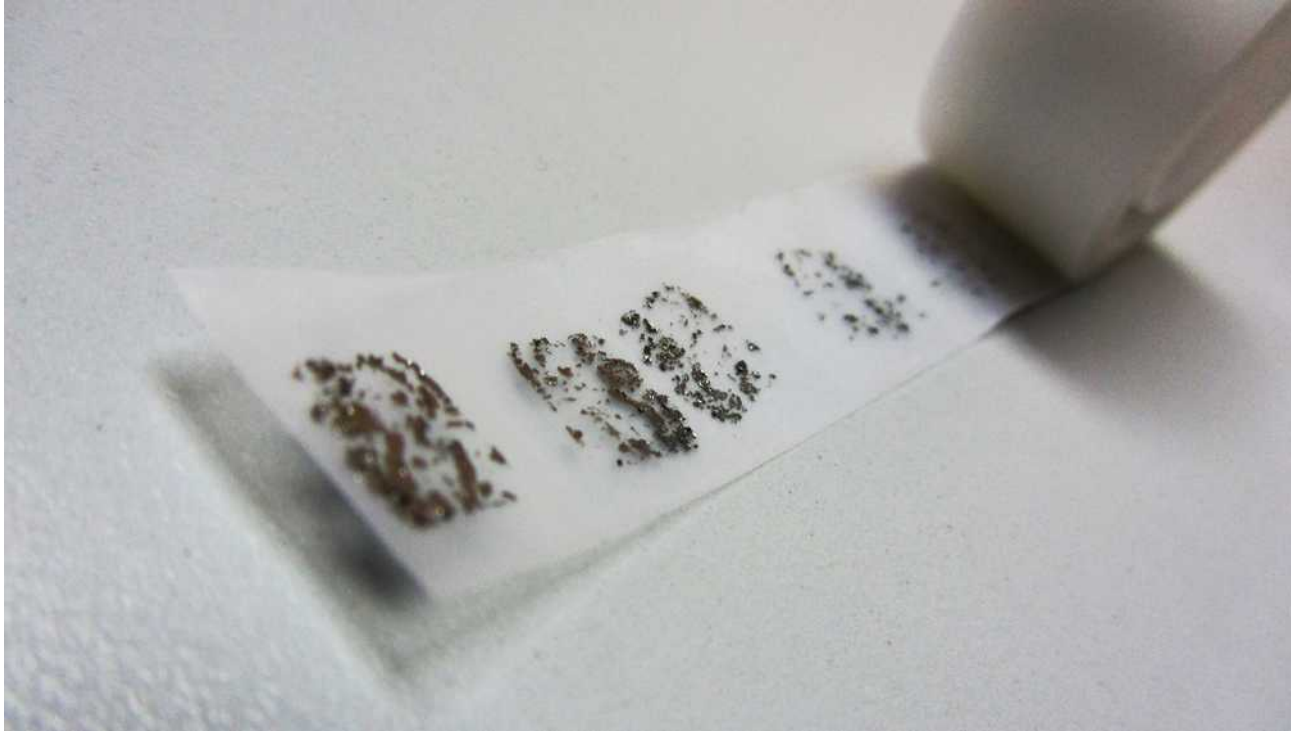
Why can the winding not change continuously? · What replaces it in two dimensions?

Next: A primer on graphene's bandstructure
the integer Quantum Hall effect



Graphene's bandstructure

Luis E. F. Foà Torres

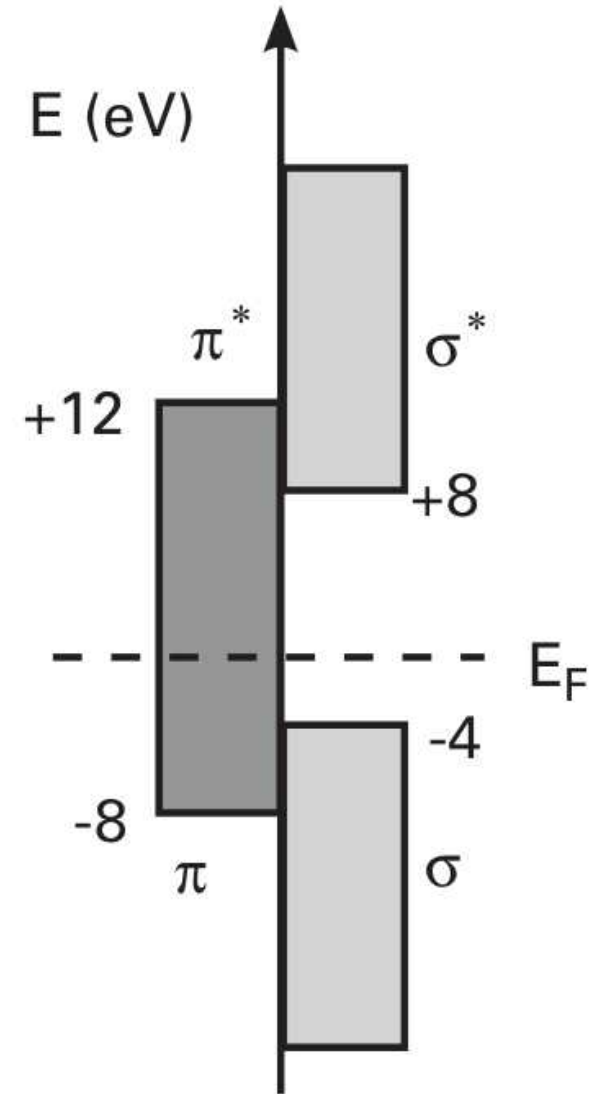
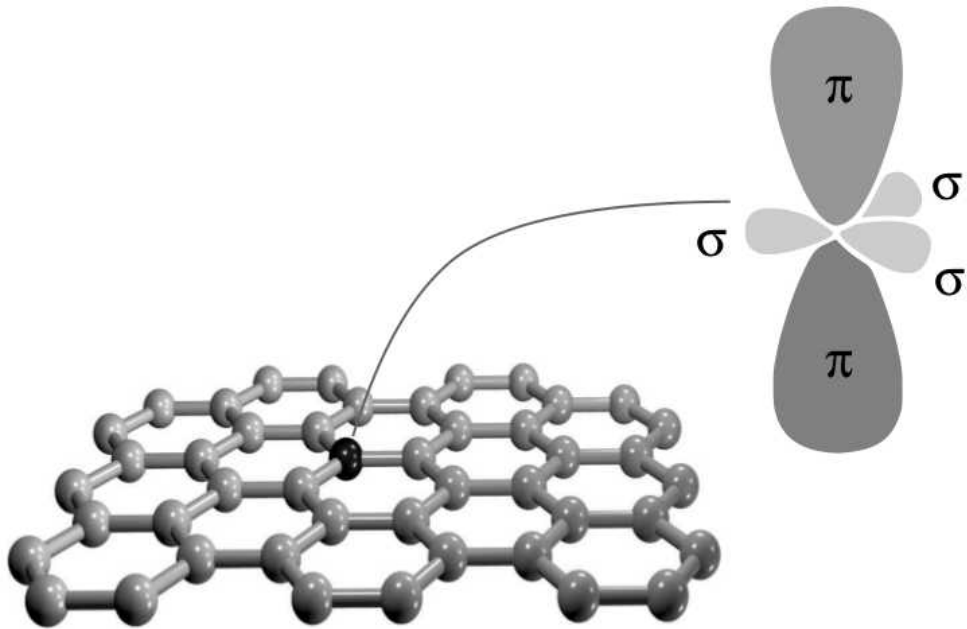


All the material from these lectures will be
here:



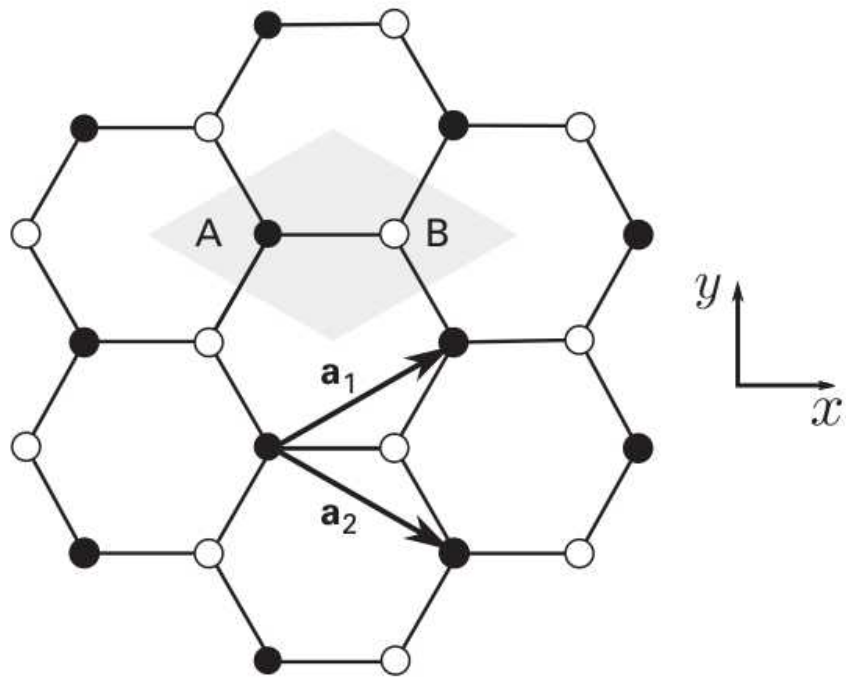
Graphene's peculiar electronic structure

Pi orbitals matter most

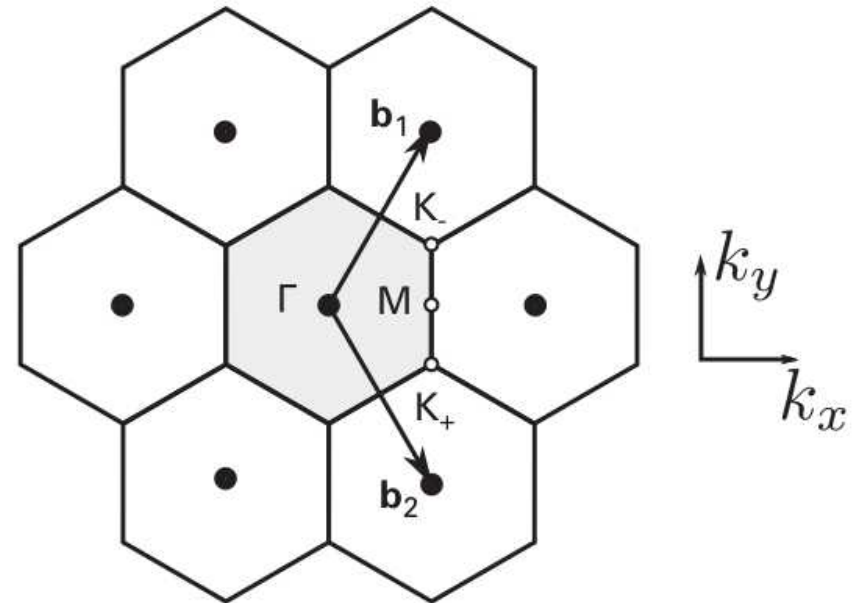


Graphene's peculiar electronic structure

Real space lattice



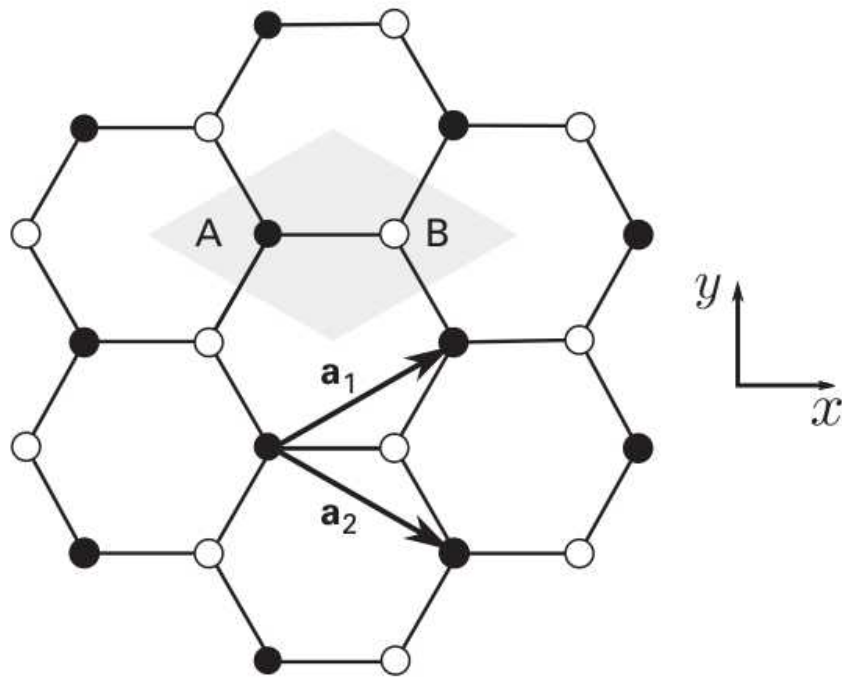
Reciprocal space



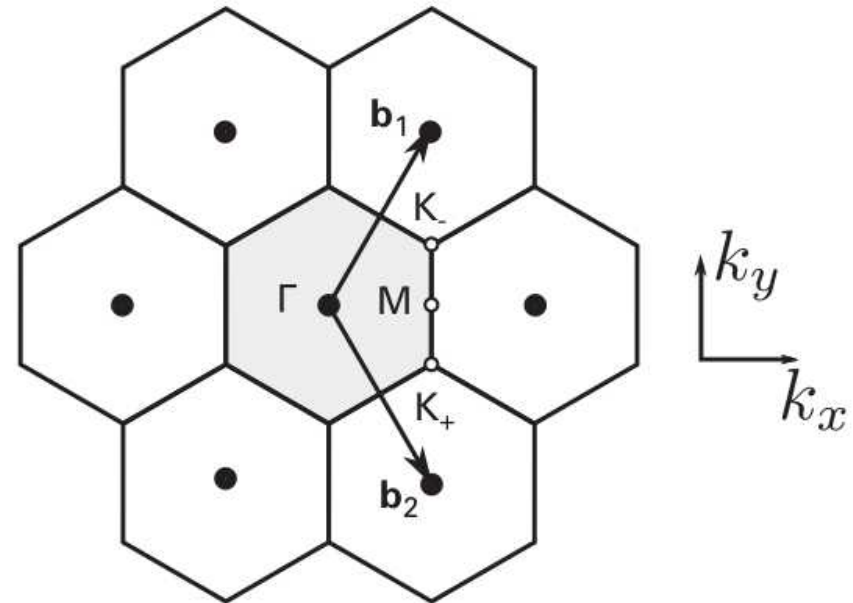
Triangular Bravais lattice with a basis

Graphene's peculiar electronic structure

Real space lattice

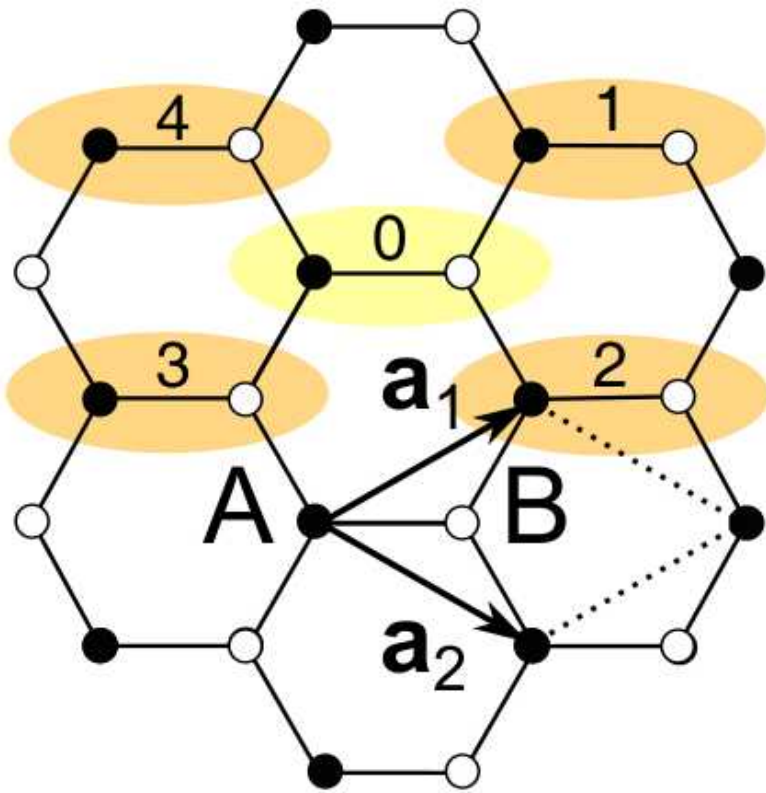


Reciprocal space



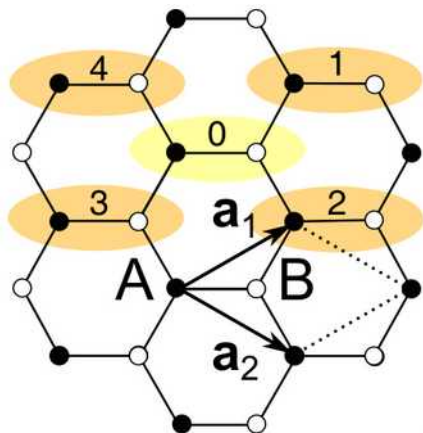
Triangular Bravais lattice with a basis

Bloch's Theorem + tight-binding approx.



1

We draw the graphene lattice highlighting a central cell and its neighbors.



We draw the graphene lattice highlighting a central cell and its neighbors.

we got two indexes:
cell index atom type

3A' 3B 4A 4B 0A 0B 1A 1B 2A 2B

2

Now we write the Hamiltonian in matrix form.
Since we have a single orbital per atom
we got a one to one correspondence between
orbitals and their space position.

Assembling the Hamiltonian is very much like playing Battleship (or "*Batalla Naval*" as is known in Spain).

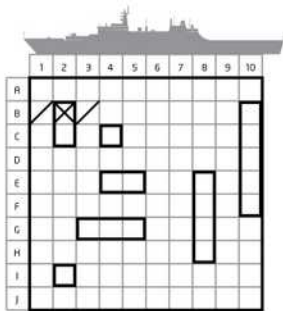


Diagram illustrating the structure of a sparse matrix, likely representing a discretized system. The matrix is partitioned into blocks along the diagonal, labeled 3A, 3B, 4A, 4B, 0A, 0B, 1A, 1B, 2A, 2B. The matrix is composed of 2x2 blocks of size 2x2. The diagonal blocks are 0A, 1A, 2A, 3A, 4A. The off-diagonal blocks are 0B, 1B, 2B, 3B, 4B. The matrix is sparse, with many zero entries. A red circle highlights the $-\gamma_0$ entry in the 0A block. A red rectangle highlights the 0B block. A black arrow points from the 0A block to the 0B block.

For example, this is the matrix element connecting sites 3B with 0A

Now we write the time-independent Schrödinger equation using Bloch's theorem for the wave-functions.

(

3 Now we write the time-independent Schrödinger equation using Bloch's theorem for the wave-functions.

$$\begin{pmatrix} \vdots \\ c_A e^{ik \cdot (R_3 - R_0)} \\ c_B e^{ik \cdot (R_3 - R_0)} \\ c_A e^{ik \cdot (R_4 - R_0)} \\ c_B e^{ik \cdot (R_4 - R_0)} \\ c_A \\ c_B \\ c_A e^{ik \cdot (R_1 - R_0)} \\ c_B e^{ik \cdot (R_1 - R_0)} \\ \vdots \end{pmatrix} = \varepsilon_k \begin{pmatrix} \vdots \\ c_A e^{ik \cdot (R_3 - R_0)} \\ c_B e^{ik \cdot (R_3 - R_0)} \\ c_A e^{ik \cdot (R_4 - R_0)} \\ c_B e^{ik \cdot (R_4 - R_0)} \\ c_A \\ c_B \\ c_A e^{ik \cdot (R_1 - R_0)} \\ c_B e^{ik \cdot (R_1 - R_0)} \\ \vdots \end{pmatrix}$$

The probability amplitude at a given site of cell j is equal to that at an equivalent site in another cell but multiplied by a phase.

4

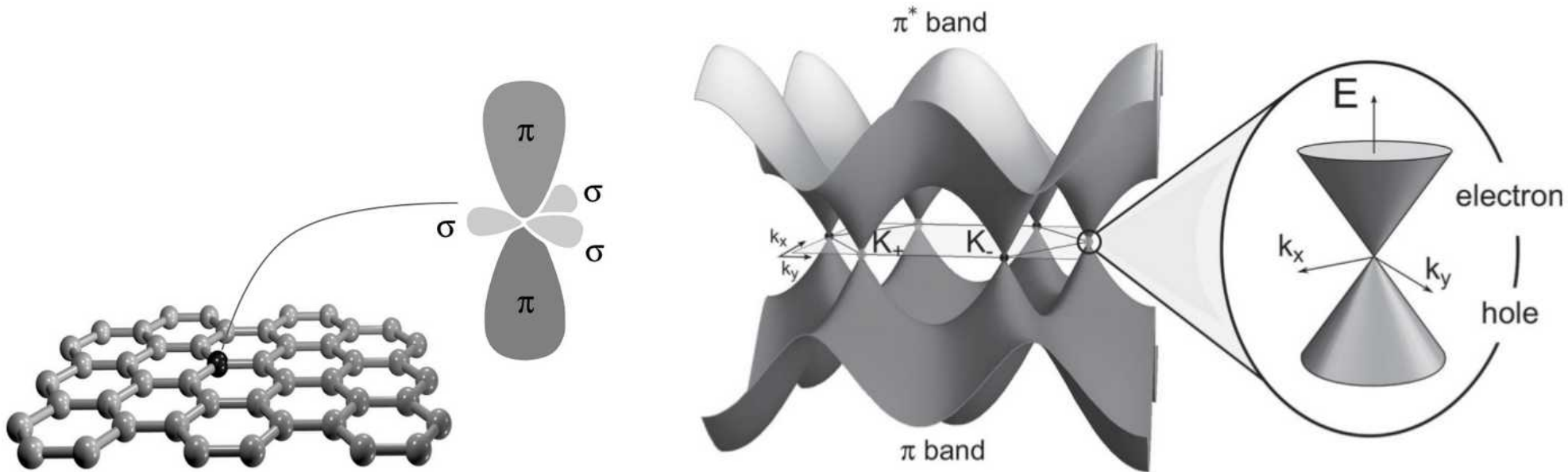
$$\begin{cases} -\gamma_0 c_B e^{ik \cdot (R_4 - R_0)} - \gamma_0 c_B e^{ik \cdot (R_3 - R_0)} - \gamma_0 c_B = \varepsilon_k c_A \\ -\gamma_0 c_A - \gamma_0 c_A e^{ik \cdot (R_1 - R_0)} - \gamma_0 c_A e^{ik \cdot (R_2 - R_0)} = \varepsilon_k c_B \end{cases}$$

$$\begin{aligned} \mathbf{R}_4 - \mathbf{R}_0 &= -\mathbf{a}_2 \\ \mathbf{R}_1 - \mathbf{R}_0 &= \mathbf{a}_1 \\ &\dots \end{aligned}$$

After some algebra one gets Eq. (2.16) for the dispersion:

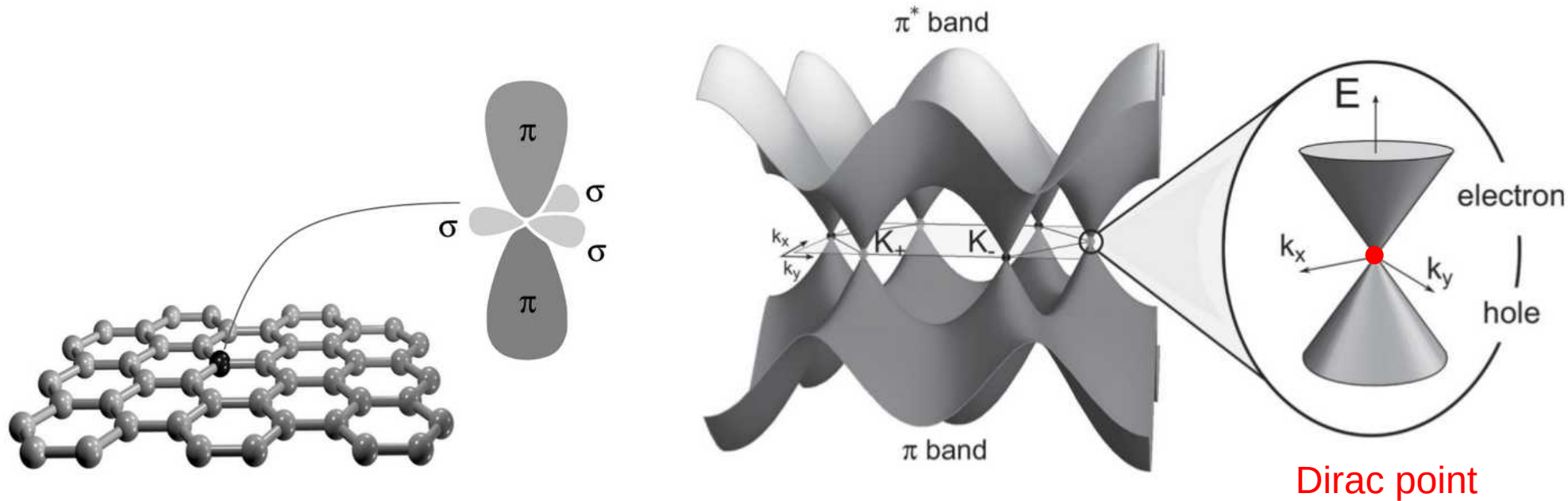
$$E_{\pm}(k_x, k_y) = \pm \gamma_0 \sqrt{1 + 4 \cos \frac{\sqrt{3} k_x a}{2} \cos \frac{k_y a}{2} + 4 \cos^2 \frac{k_y a}{2}}.$$

Bandstructure and low-energy approximation



Technically, a zero-bandgap semiconductor.

Bandstructure and low-energy approximation

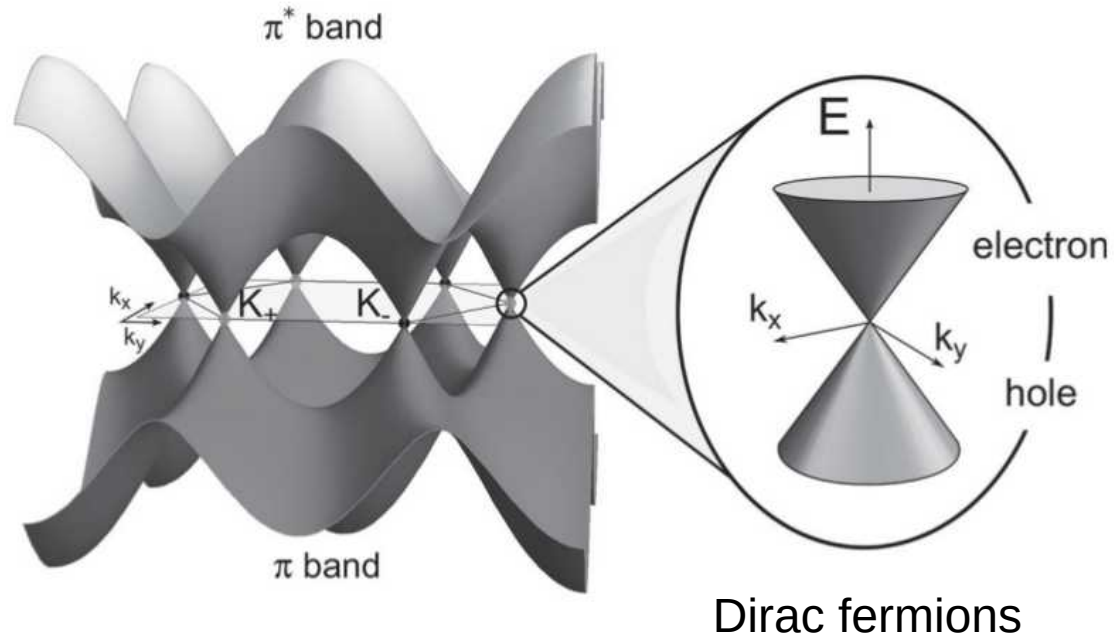
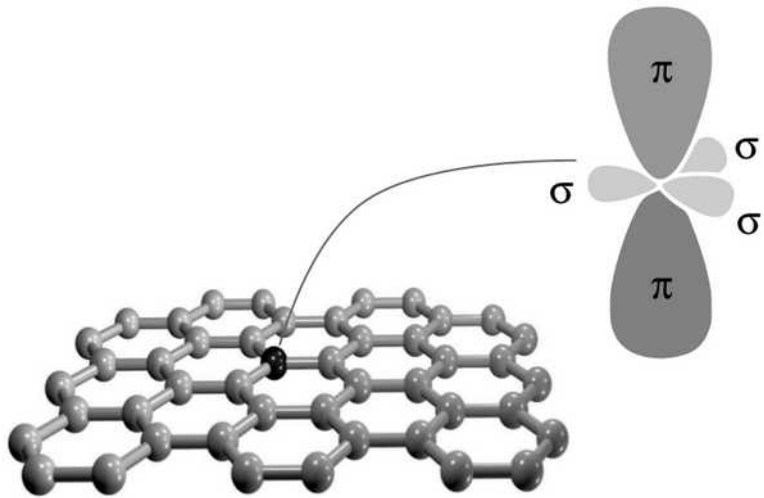


Dirac point is not accidental,
it's protected by symmetries.

Time-reversal (broken by a magnetic field)
and inversion symmetry (broken by a potential which distinguishes A from B-type atoms).

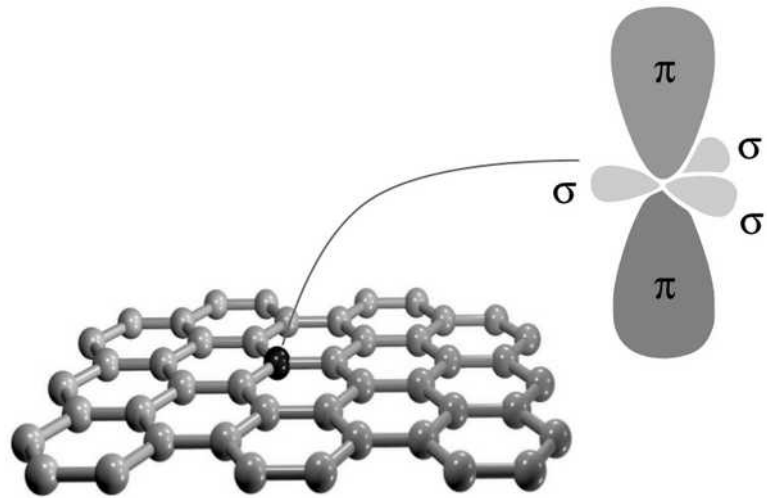
See Exercises 2.2 and 2.16

Bandstructure and low-energy approximation



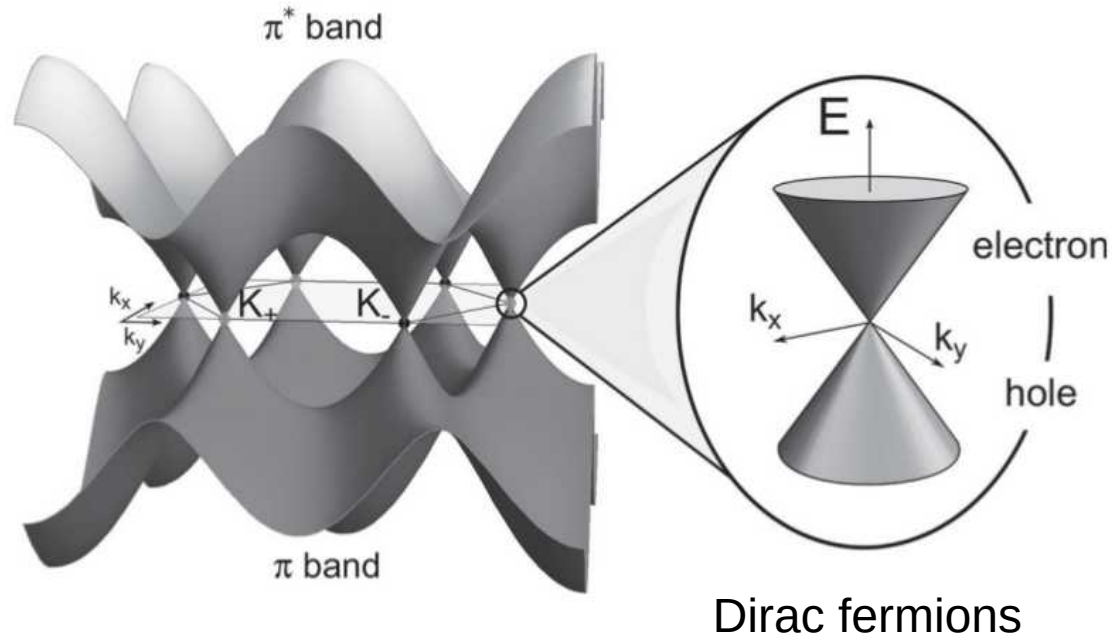
$$\mathcal{H}_{K_+} = \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix}$$

Bandstructure and low-energy approximation



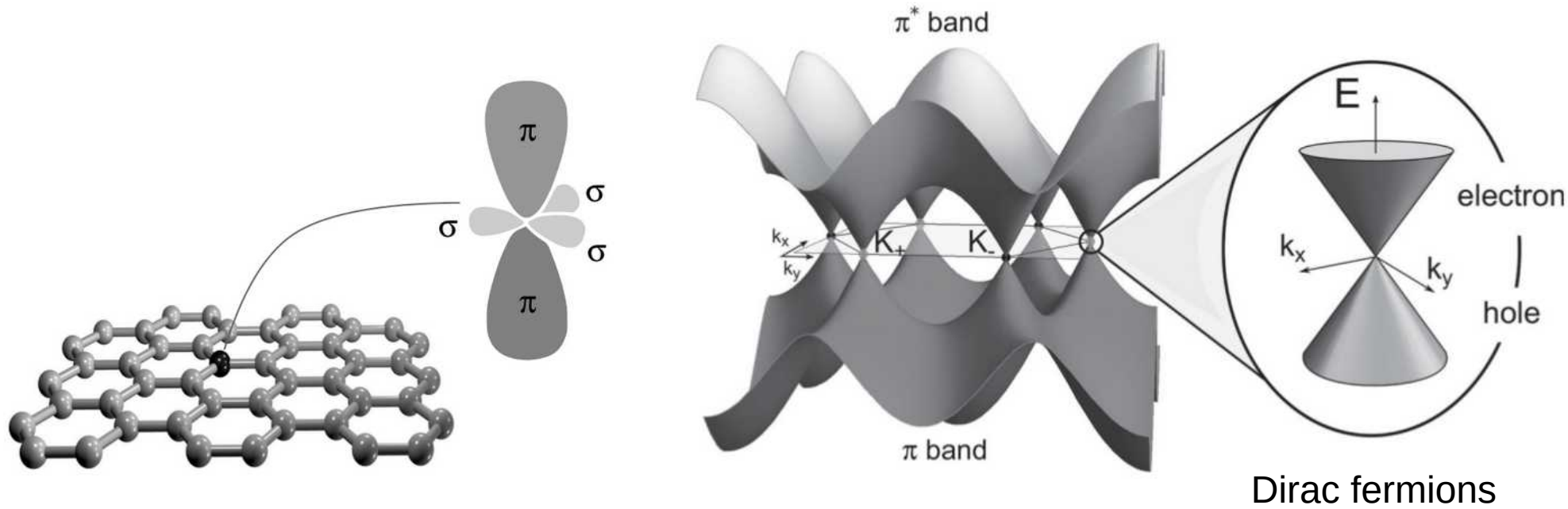
Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$\begin{aligned} \mathcal{H}_{K_+} &= \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix} \\ &= v_F(p_x \sigma_x + p_y \sigma_y), \end{aligned}$$

Bandstructure and low-energy approximation

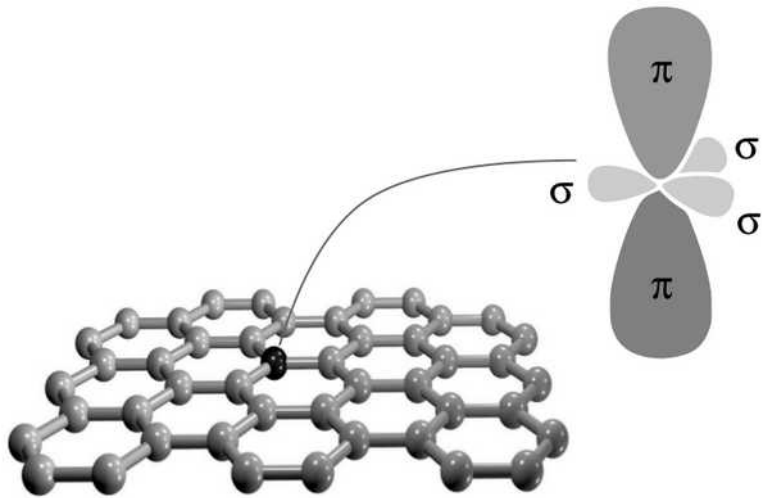


$$\mathcal{H}_{K_+} = \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix}$$

$$= v_F(p_x \sigma_x + p_y \sigma_y),$$

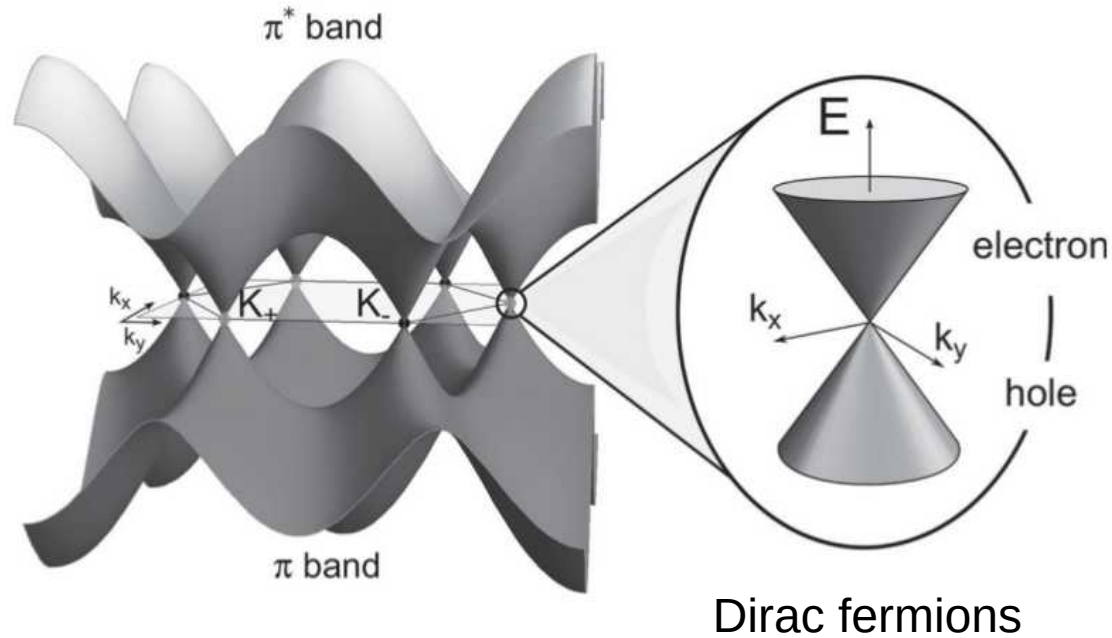
$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \quad \mathcal{H}_{K_+} = v_F \hat{\sigma} \cdot \mathbf{p},$$

Bandstructure and low-energy approximation



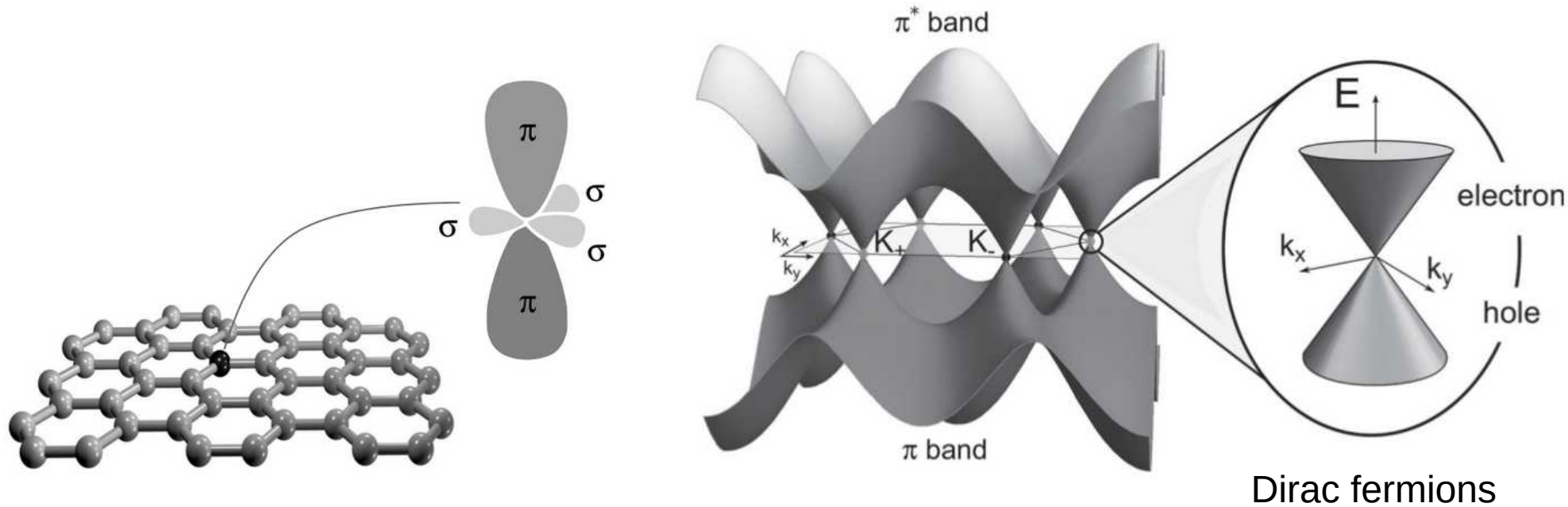
The state-vector has
a spinor structure

$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$



$$\begin{aligned}\mathcal{H}_{K_+} &= \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix} \\ &= v_F(p_x \sigma_x + p_y \sigma_y), \\ \mathcal{H}_{K_+} &= v_F \hat{\sigma} \cdot \mathbf{p},\end{aligned}$$

Bandstructure and low-energy approximation



Relativistic-like particles with
ZERO REST MASS but “low”
speed

$$v_F = 10^6 \text{ m/s} \pm 5\%$$

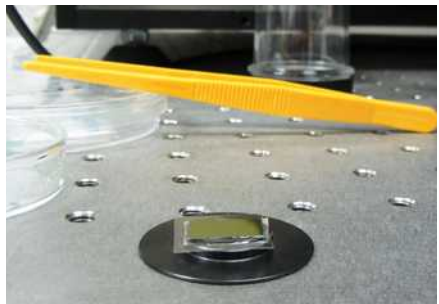
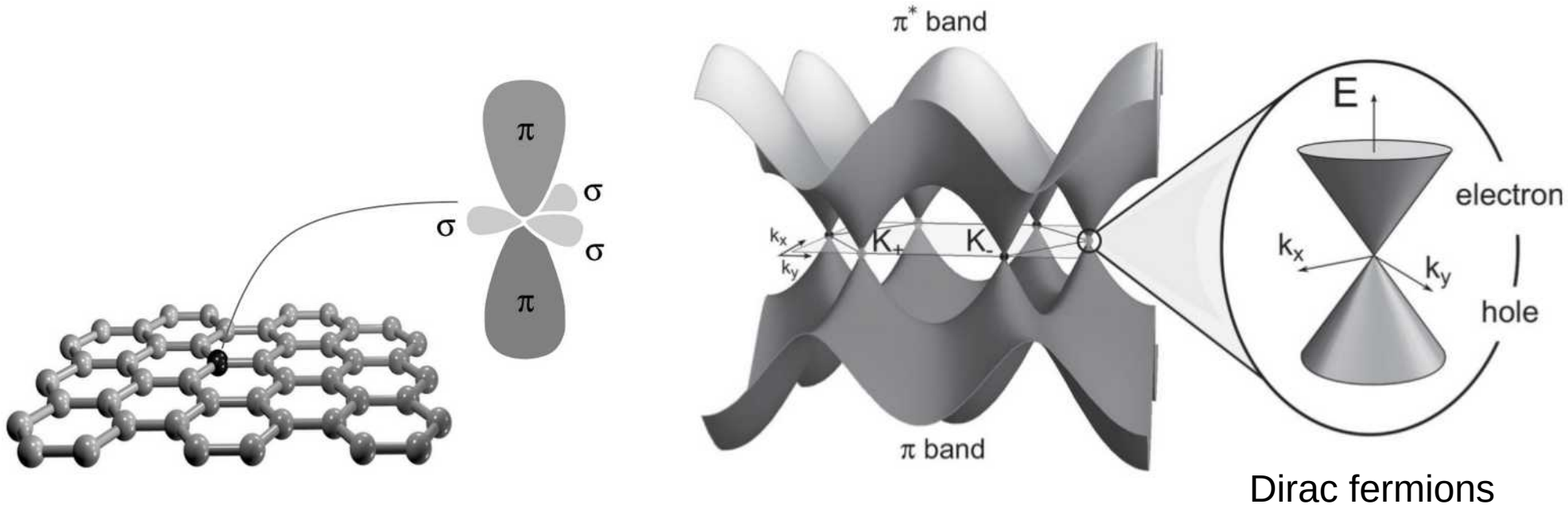
$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\mathcal{H}_{K_+} = \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix}$$

$$= v_F(p_x \sigma_x + p_y \sigma_y),$$

$$\mathcal{H}_{K_+} = v_F \hat{\sigma} \cdot \mathbf{p},$$

Bandstructure and low-energy approximation



“CERN on a Desk”
A. Geim

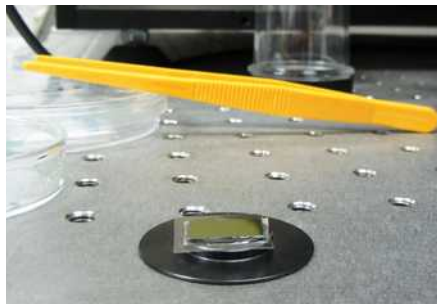
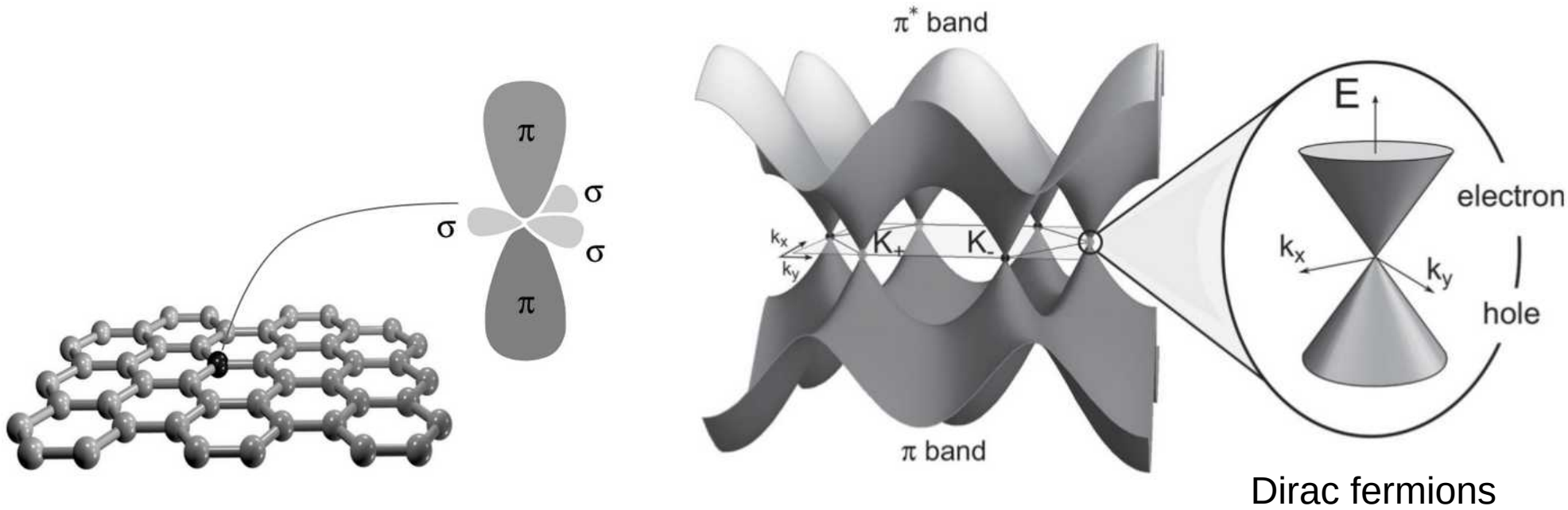
$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\mathcal{H}_{K_+} = \hbar v_F \begin{pmatrix} 0 & k_x - ik_y \\ k_x + ik_y & 0 \end{pmatrix}$$

$$= v_F(p_x \sigma_x + p_y \sigma_y),$$

$$\mathcal{H}_{K_+} = v_F \hat{\sigma} \cdot \mathbf{p},$$

Bandstructure and low-energy approximation



“CERN on a Desk”
A. Geim

$$\hat{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

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$$\mathcal{H}_{K_+} = v_F \hat{\sigma} \cdot \mathbf{p},$$

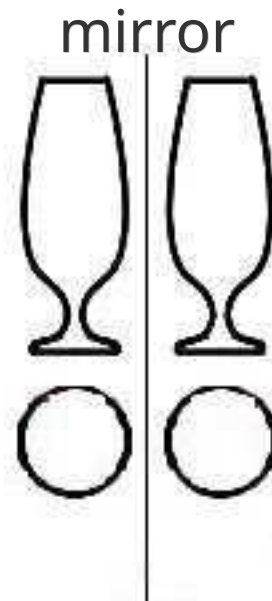
Chirality and pseudospin

Chirality and pseudospin

An object that cannot be superimposed on its mirror image is called chiral



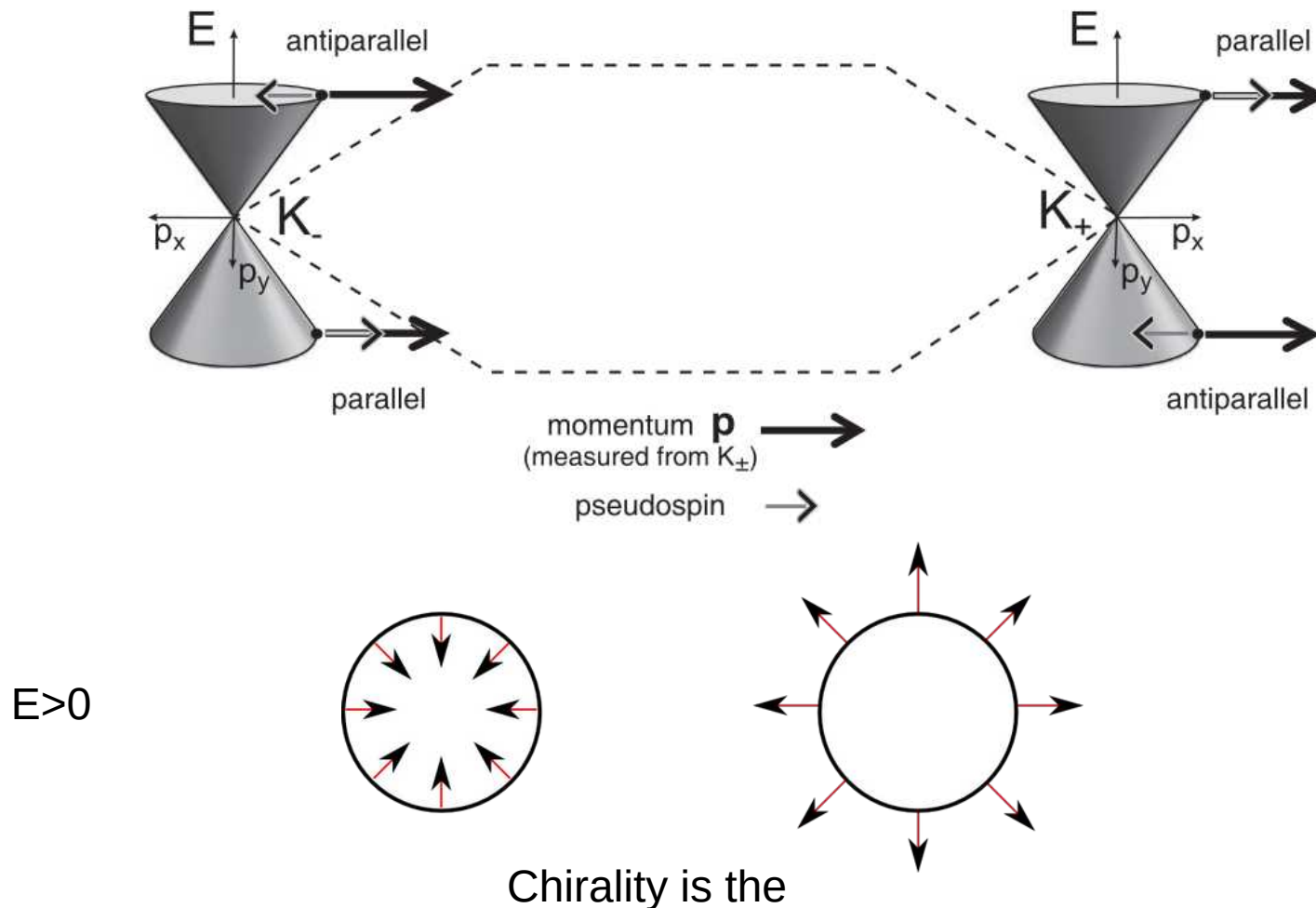
chiral
objects



non-chiral
objects

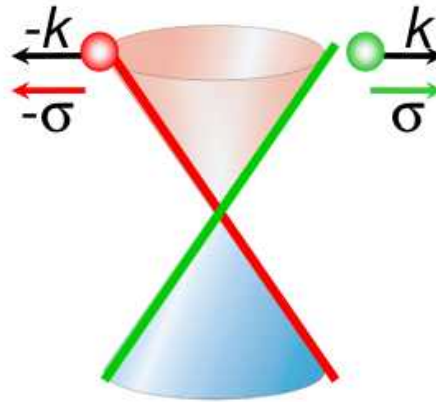
Chirality and pseudospin

The low-energy Hamiltonians around each valley are not the same, they are transposed (TRS).



See Exercise 2.13 (online)

Klein tunneling (absence of backscattering)



Electrons moving to the right

Electrons moving to the left

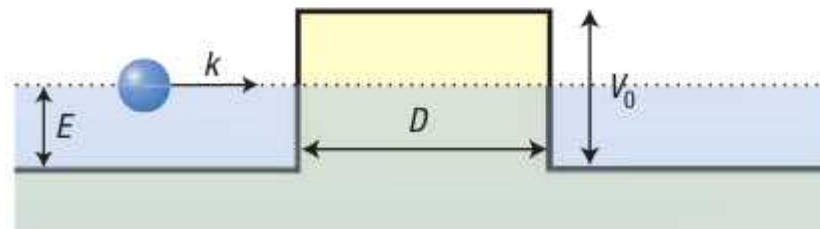
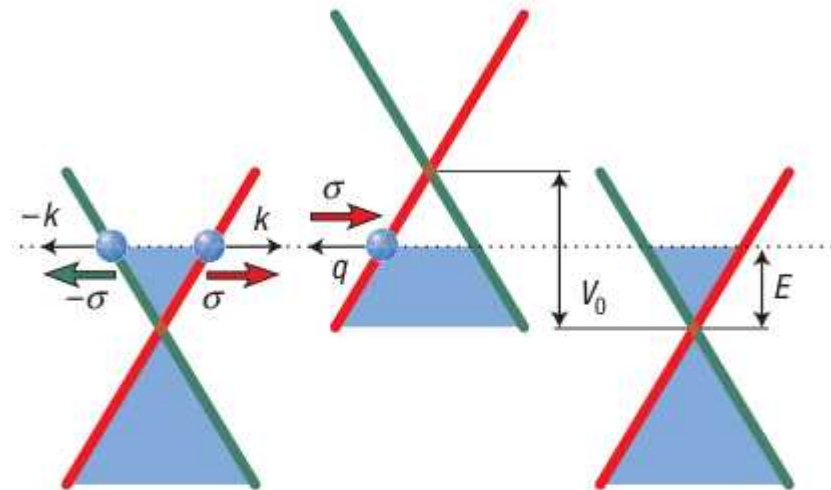
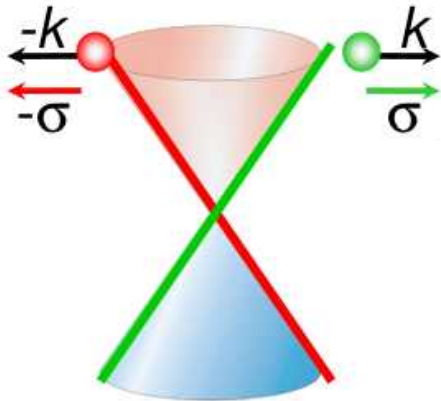
pseudospin

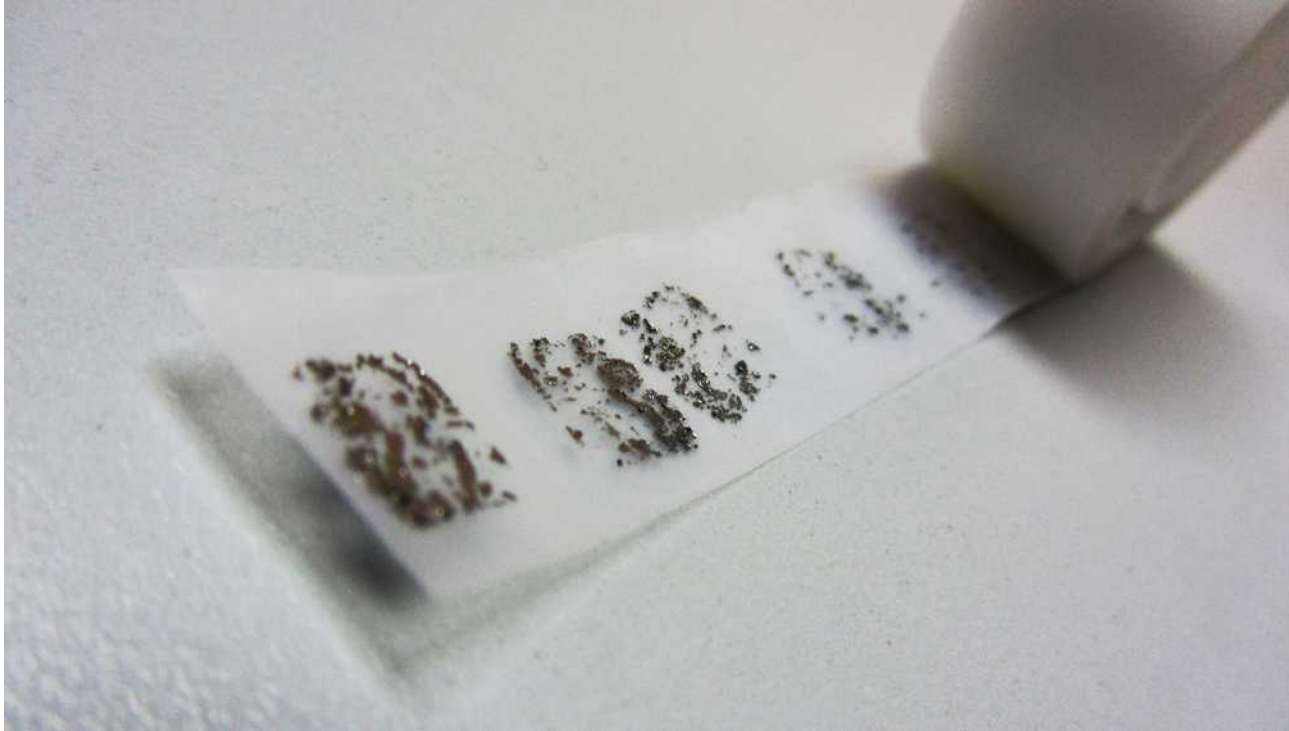
σ

$-\sigma$

Klein tunneling (absence of backscattering)

Scattering backwards would require changing (“flipping”) the pseudospin
→ Absence of backscattering!





All the material from these lectures will be here:

